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Objectives

Adults Learning Mathematics (ALM) – An International Research Forum has been established since 1994 (see www.alm-online.net), with an annual conference and newsletters for members. ALM is an international research forum that brings together researchers and practitioners in adult mathematics/ numeracy teaching and learning in order to promote the learning of mathematics by adults. Since 2000, ALM has been a Company Limited by Guarantee (No.3901346) and a National and Overseas Worldwide Charity under English and Welsh Law (No.1079462). Through the annual ALM conference proceedings and the work of individual members, an enormous contribution has been made to making available research and theories in a field which remains under-researched and under-theorized. In 2005, ALM launched an international journal dedicated to advancing the field of adult mathematics teaching and learning.

Adults Learning Mathematics – An International Journal is an international refereed journal that aims to provide a forum for the online publication of high quality research on the teaching and learning, knowledge and uses of numeracy/mathematics to adults at all levels in a variety of educational sectors. Submitted papers should normally be of interest to an international readership. Contributions focus on issues in the following areas:

- Research and theoretical perspectives in the area of adults learning mathematics/numeracy
- Debate on special issues in the area of adults learning mathematics/numeracy
- Practice: critical analysis of course materials and tasks, policy developments in curriculum and assessment, or data from large-scale tests, nationally and internationally.

The ALM International Journal is published twice a year.

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In this Volume 14(1)

Editorial

This issue of Adults Learning Mathematics: An International Journal is the first for 2019. It includes four articles; two were presented at the ALM-25 conference in London, and two were submitted separately to the Journal. The articles exemplify the wide range of topics and contexts that are of interest to the adults learning mathematics community. The topics involve specific mathematical ideas such as proportional reasoning, pedagogical matters such as the use of self-study strategies, and broader concerns such as prior experiences and feelings towards mathematics. The articles discuss individuals taking part in prison education, adults (and young people) returning to learning in community settings and those studying mathematics as a service subject in university.

Linda Ahl provides an analysis of a test designed to detect students' prior understanding of proportional reasoning. She presents the results of a survey involving 32 adults from a Swedish prison education program. The test was designed by drawing on previous published research papers, including items that have already been tested successfully. Specific mathematical contents are covered with the test, including additive vs. multiplicative reasoning using proportions, algebraic rules for fractions as part/part ratios, and part/whole fractions and proportions, different representations for proportions, and scale working with 2D and 3D objects. She draws on three different cases: two men for whom several years have passed since they studied mathematics, one of them being placed in a special group for students with disabilities, plus another man who clearly had a negative school experience, especially with mathematics. The data collected seems to support the test as a good instrument to detect adults' prior knowledge on proportional reasoning. This article, including both conceptual analysis and empirical work makes a valuable contribution to our knowledge of how adults use proportional reasoning in different contexts.

Pragati Jain and Merran Rogers discuss the need for and development of a numeracy course for those in need of further preparation for entry to university-level education. Rather than replicating secondary school mathematics instruction, the semester-long Academic Numeracy unit focuses on the critical thinking that is required for making sense of mathematical information a university and in everyday life. Guided by Watson's (2009) "Four Resource Model for Critical Numeracy," the course is structured around investigations of real world topics of interest such as Climate Change and Mandatory Bike Helmet legislation, and uses a statistical focus to guide analysis of information. Through these investigations, learners develop statistical literacy and data analysis experience, proportional and algebraic reasoning and financial numeracy. The article describes the ways non-traditional learners, with diverse needs and skills, are provided with self-study resources, tutorials and mini-quizzes to introduce or reinforce mathematical content necessary for their understanding and statistical analyses. Since the Academic Numeracy unit was so different from their previous mathematics classes, students were initially surprised and challenged, but by the end of the course, students reported high levels of satisfaction and learning.

Maria Ryan, Olivia Fitzmaurice and Patrick Johnson report on the ways that 20 adult learners described their experiences with mathematics across their life spans while engaged in pre-tertiary mathematics courses at an Irish University. Individually, the adults reflected on their own experiences from early and later schooling, from their post-school lives, their decision to return to education, and their expectations of the role of mathematics in their future. The learners then represented the journey with a personal theme or metaphor explaining its meaning. The various themes are categorised and their potential influence over current learning is considered. The relationships between personal themes, demographic characteristics and scores on an assessment of mathematics anxiety are explored. From this approach we see how the title of this interesting paper was generated: themes from three students are reported on and involve "divorce, evil and (a) regime of terror". The authors note the importance to the adult learner in talking about these past experiences in order to heal the “scars left behind” and be ready to succeed. The process was also thought to help university staff in understanding how best to support the development of these learners.

Gail FitzSimons provides a wide-ranging overview of many issues in adult mathematics education, in a paper based on her Plenary Address to the ALM-25 conference in London last year. She highlights the crucial importance of developing a depth of mathematical knowledge / understanding, as contrasted with the ‘learning outcomes’ approach of many contemporary courses. In considering what are adults’ interests in learning mathematics, she argues that, by 2020, the top three skills needed by workers will be: complex problem-solving, critical thinking and creativity. She offers many examples from doing mathematics at work (paid and unpaid) and draws on Basil Bernstein’s advocacy of three pedagogic rights which enable democracy to function: enhancement, inclusion and participation. Gail concludes with a case study of the measurement learning outcome and a contrasting ‘depth-understanding’ approach in a numeracy programme for young learners returning to study mathematics.

This is the final issue of ALM International Journal that will be produced under Javier Díez-Palomar as Editor-in-Chief. Javi took on this responsibility beginning with volume 6(1), which came out in February 2011. Under his leadership the Journal has developed and thrived: we have moved to a normal production of two issues a year, and we have reviewed our editorial procedures to ensure that the standard of articles will continue to improve. From ALM-17 in Rotterdam, we have moved to a renewed Editorial Team, and refreshed the Editorial Board. These changes are still ongoing.

From the next issue, volume 14 (2), to appear later this year, the Editor-in-Chief’s duties will be shared among Jeff Evans, Linda Galligan, and Lynda Ginsburg. We hope you will find the articles in this issue interesting and thought-provoking, and we encourage you to feed back your impressions, and to consider contributing your articles to the Journal.

The Editorial Team

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Designing a research-based detection test for eliciting students' prior understanding on proportional reasoning

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Abstract

In the Swedish Prison Education Program only two out of ten reach a passing grade in their mathematics courses. Large variation in prior knowledge makes it difficult to meet the students at their level. This paper reports on a project aiming to enhance students' possibilities to access mathematics through individualization. Research findings on the development of the pervasive mathematical idea of proportional reasoning are used to construct a test on proportional reasoning, designed to work specifically with students with large variation in prior knowledge. The test presented here, combined with a follow-up clinical interview, can be used in adult education in general as a basis for individualizing instruction.

Key words: proportional reasoning; adult education; prison; individualized instruction

Introduction

Already in 1968, Freudenthal drew attention to the inequity of an unbalanced mathematics education system that provides mathematics education to only a small group of citizens who "fit" within the system. He proposed that instead of worrying that the highest level may be too low for the gifted students, we should strive for educational justice for all people in society. Educational injustice is particularly prominent in mathematics (Stinson, 2004). The illusion that mathematics is only for people with special gifts has lived on for at least 2300 years, as illustrated by Plato's claim that "We shall persuade those who are to perform high functions in the city to undertake calculation, but not as amateurs." (Plato, trans. 1996, p. 219). In our modern society, mathematics acts as a key for passing through the gates to economic access, full citizenship, and higher education (Stinson, 2004). This gatekeeping function makes equitable access to mathematics education a question of democracy. To enhance democracy, mathematics needs to be an inclusive instrument for empowerment rather than an exclusive instrument for stratification. Instead, in most school systems, some students become disqualified from mathematics while others qualify (Jurdak et al., 2016; Popkewitz, 2002). When these disqualified students return to mathematics as adults we cannot expect success by repeating educational practices that did not work in their past. Therefore, an equitable access to mathematics education for such a disadvantaged group needs to acknowledge not only prior knowledge, but also life conditions alongside personal study narratives.

The Swedish prison education program is an educational environment where students with without an upper secondary diploma cluster. Close to 50 % of the prisoners lacks an upper secondary diploma (Riksrevsionen, 2015). In the prison education program a second chance to obtain this diploma is provided. Nevertheless, only two out of ten students pass their mathematics courses. This is disturbing in itself, but particularly in light of the resources available. The teachers are university trained upper secondary school mathematics teachers and the students sign up voluntarily. These conditions should ensure a high potential for effectiveness. However, the student group shows significant variation in age, ethnicity, socioeconomic and school backgrounds and life experience in general. These factors may account for such low pass rates, and justifies, a plausible reason to focus on such individualized differences.

It is an unchallenged claim that teaching accounts for an individual's prior knowledge, his [1] life conditions and his rationale for studying mathematics have better prospects to succeed than a “one size fits all” teaching practice. When teaching adults returning to mathematics, such considerations are of particular importance. Given that a teaching approach aiming at individualized instruction [2] requires information on the individual's prior knowledge, his life conditions and his rationale for studying mathematics, teachers need strategies to collect the necessary data. This paper focuses on a project aiming to elicit one piece of information: the students’ prior knowledge of mathematics. The setting is the special context of adults studying mathematics in prison. While students in lower- and upper secondary school can be screened on the content taught last year, as a means to provide a picture of the current state, adults with many years away from school possess far more unpredictable knowledge. Therefore, adults returning to mathematics after several years away from formal schooling present a particular challenge. A prerequisite for individualization of courses is that teachers have the opportunity to find out students' prior mathematical experience and their mathematical competencies and adapt their teaching accordingly. Realizing this opportunity hinges on the teachers’ competencies, e.g., they need to action their teaching competency more effectively (Niss & Højgaard, 2011). Further, it is reasonable to believe that teachers’ practical ability to investigate students’ prior knowledge increase if they are provided with support. They may need a test to investigate the students’ prior knowledge, to support teachers’ individualization process.

The test needs to fulfill certain properties. It needs to give information on students’ prior knowledge both vertically, in relation to progress throughout school years, and horizontally throughout taught topics in compulsory school. A full investigation would be far too time-consuming, but since proportionality may be the most important, pervasive and powerful idea in elementary school mathematics (Behr et al., 1992; Hilton, Hilton, Dole, & Goos, 2013; Karplus, Pulos, & Stage, 1983; Lamon, 2007; Sowder et al., 1998), proportional reasoning was chosen as the basis for a test combined with a follow-up clinical interview (Ginsburg, 1981). Furthermore, the test has to give information that discriminates among students. Discovering that all students suffer from the same lack of reasoning skills will not help teachers to individualize instruction. We want to discover students’ difficulties as well as be provided with information on how students tackle known thresholds for the development of proportional reasoning. By implementing results from the extensive research on the topic, we can relate students’ results to the learning goals and core content in the Swedish national curricula. Therefore, students’ ability to reason proportionally can provide an access point for individualized instruction. The aim of this study is to investigate if it is possible to design a test that together with a clinical interview gives information on students’ prior knowledge on proportional reasoning so as:

- discriminate among students and,
- provide teachers with an access point for designing individualized instruction.

Theoretical underpinnings for the development of a test

Mathematical reasoning is essential for doing mathematics. It is one of eight competencies for identifying and analyzing students’ mathematical knowledge, described in the Danish KOM-project (Niss & Højgaard, 2011). “The mathematical reasoning competency consists, first, of the ability to follow and assess mathematical reasoning, i.e., a chain of arguments put forward – orally or in writing – in support of a claim.” (Jankvist & Niss, 2015, p. 264). The kind of mathematical reasoning called proportional reasoning is a prerequisite for successful further studies in mathematics and science, since multiplicative relations underpin almost all number-related concepts studied in elementary school (Behr et al., 1992; Lamon, 2007). A proportion is defined as a statement of equality of two ratios $a/b = c/d$. Proportion can also be defined as a function with the isomorphic properties $f(x+y) = f(x) + f(y)$ and $f(ax) = af(x)$ (Vergnaud, 2009). A function, $A(x,y)$, can also be linear with respect to several variables, (n-linear) functions. For example the area functions for a rectangle with sides x and y is bilinear (2-linear) since $A(x,y) = xy$ and it is easy to check that this function is linear with respect to each of its variables when the other is considered constant.

Proportionality is a key concept in mathematics education from elementary school to university (Lamon, 2007). Despite the pervasive nature of proportional reasoning throughout the school years it is well known that children around the world have considerable difficulty in developing the mathematical competency to reason about fractions, percentages, ratio, proportion, scaling, rates, similarity, trigonometry, and rates of change (Behr, Harel, Post, & Lesh, 1992; Lamon, 2007). Typically, proportional reasoning problems come in the shape of a missing value problems or comparison problems (Lamon, 2007). In the former, a multiplicative relation is present where three elements are provided and the fourth is to be found. The latter asks the student to compare which ratio is the bigger or smaller.

Key points for the development of proportional reasoning and the building of multiplicative structures can be identified from the accumulated research (c.f. Behr, Harel, Post, & Lesh, 1992; Fernández et al., 2012; Lamon, 2007; Shield & Dole, 2013; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010; Vergnaud, 1983). To illustrate the conceptual structure of the proportionality concept, four central key points are presented below.

Key point 1. Students need to be able to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present. The ability to distinguish additive from multiplicative comparisons constitute a major stumbling block for students (Van Dooren et al., 2005). Students need to be able to recognize that a proportional situation exists when the comparison is multiplicative (Shield & Dole, 2013). In Sweden, students get acquainted with additive strategies for reasoning about quantities in grades 4 to 6. For example, an increase in price by 10% can be calculated in two steps. First, calculate how much 10% is and then add this to the original price. A transition from an additive to multiplicative thinking approach is introduced in grades 7 to 9. The new price can now be approached in one multiplicative step: the original price multiplied by the factor 1.1, to find the new price. The error of using additive strategies on proportional situations increases during primary school and decreases during secondary school (Fernández et al., 2012). A desirable development in students' reasoning would be that they, after being introduced to multiplicative reasoning, still hold on to their ability to use additive strategies when appropriate. However, research findings show that once students have been introduced to multiplicative strategies they tend to overuse this approach on everything that resembles a proportional situation (Van Dooren et al., 2005). Further, non-integer ratios cause more errors than integer ratios (Fernandez et al., 2012; Gläser & Riegler, 2015), while the non-integer situations can be considered to require a more developed knowledge of rational numbers.

Key point 2. Students need to be able to draw connections to the algebraic rules for fractions when working with part/part ratios, part/whole fractions and proportions, $a:b = c:d$. Many situations require that students can relate to part/part ratios and part/whole fractions (Vergnaud, 1983). For example, if a company employs 11 women and 31 men, the part/whole fractions 11/42 and 31/42 represent the relation of women and men related to the whole. If asked to determine the company's gender distribution, it is instead the part/part ratio 11:31 between women and men that is relevant. When a ratio connects two parts of the same whole, students may not adequately recognize the difference between part/part and part/whole relationships (Clark, Berenson, & Cavey, 2003). It is not easy for students to approach situations that require shifting from part/part to part/whole situations. Moreover, students need to connect mathematical ideas. Since ratios can be written in fraction form, they obey the same mathematical laws as fractions (Shield & Dole, 2002).

Key point 3. Students need to recognize and use a range of concrete representations for proportions, e.g., tables, graphs, formulas and drawing pictures. Students tend to apply linear proportional reasoning on scaling in two- and three dimensions, without considering the nature of the item. Van Dooren et al. (2010) found that students tend to use linear proportional reasoning even when it is inappropriate e.g., in word problems where a real word context is required to solve the problem. For example: Farmer Gus needs 8 hours to fertilize a square pasture with sides of 200 meters. Approximately how much time will he need to fertilize a square pasture with sides of 600 meters? Recognizing this as a missing value problem i.e., three values given and one unknown, this problem will trigger a cross-multiplication type solution which gives the wrong answer of 24 hours. Since scale is one of the major themes that span mathematics, chemistry, physics, earth/space science and biology it is crucial for students to gain knowledge of the concept of scale. Scale in one, two, and three dimensions is a central unifying concept that crosses the science domains, crucial for understanding science phenomena (Taylor & Jones, 2009).

Key point 4. Students need to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity. Proportionalities can be represented in different ways, e.g., with words, pictures, algebraically, with graphs or tables. Shield and Dole (2013) enhance the use of a range of representations to promote students' learning. If students are given the opportunity to work with graphs, tables and other diagrams that illustrate the proportional situation present in the mathematical task, their conceptual knowledge is promoted (Vergnaud, 2009). Further, their ability to see connections between problems that are based on the same mathematical idea is enhanced, e.g. to see that missing value problems on similarity, proportional functions and speed problems can be illustrated with different representations but approached with the same mathematical idea.

Several concepts are in play when students reason with proportional quantities. The set of intertwined concepts required for the development of proportional reasoning constitutes a *conceptual field* (Vergnaud, 2009). A conceptual field is a set of situations and concepts tied together. As the theory of conceptual fields show, together with other well-known theoretical frameworks for conceptual knowledge, the meaning of a single concept does not come from one situation only (Sfard, 1991; Tall & Vinner, 1981; Vergnaud, 2009) but from a variety of situations demanding mathematical reasoning related to the concept in question. The conceptual field of intertwined concepts in play in proportional reasoning cover at the least "linear and *n*-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division" (Vergnaud, 1983, p. 141). It is the complexity of the concepts in play together with the pervasive nature of proportional reasoning from elementary school to university that makes proportional reasoning suitable for the design of a test.

Methodology

The test was developed and tried out in the context of the Swedish prison education program, where approximately 650 adult students take mathematics courses each year. The vast majority, 80 %, study basic courses where proportional reasoning spans across all core content in the national mathematics curricula topics: *Understanding and use of numbers, Algebra, Geometry, Probability and statistics, Relationship and changes* [3]. The content given in the national curricula for school years 3, 6 and 9 indicate a clear progression of proportional reasoning. And correspondingly, an analysis of the national assessment given in grade 9 in the year 2013 showed that 69% of the items could be solved with applying proportional reasoning. The basic mathematics course for adults, corresponding to the 9 years of Swedish elementary school, is divided into four parts. By the end of part four the students are supposed to master: methods for calculating with rational numbers, fractions and ratios; the concept of a variable; formulas; methods for equation solving; scale in one, two and three dimensions; similarity; and uniform probability. When students pass the fourth part they get a grade for the entire course. Thus opportunities for individualization open up. Depending on students' prior knowledge they can start on different levels in the elementary course. Hence, this opportunity supports the need for a test that discriminates among students' and provides teachers with an access point for designing individualized instruction.

An important design choice for the test was to use a multiple-choice design. Even though open response tests are a powerful method to elicit students' knowledge, the advantages of multiple-choice tests were in this case considered to be the best option. An open response test can be a negative experience for students with low prior knowledge, since they may be unable to supply any answers. If possible, we want to avoid negative experiences in the beginning of a mathematics course. A multiple-choice test is easy to take for the students. Even when they do not have the mathematical competencies to reason and solve an item, they can still provide an answer by intuition or chance. Since multiple-choice tests "may be designed so that wrong answers clearly indicate well-known misconceptions or specific incorrect strategies of solution" (Ginsburg, 2009, p. 113), common errors, known from accumulated prior research and experience, are incorporated among the answer alternatives.

The items in the test were chosen from published research papers, with the intention to draw on knowledge from the research field on proportional reasoning. The rationale for my choices is as follows: a) the items have already been proved to work well for giving information on students' knowledge, and b) extensive background information of the nature of the mathematical reasoning in play are provided as well as analyzes of students results. Referring to the key points presented in the theory section, the potential reasoning related to each item involves several concepts and competencies, yet the items can still be categorized as referring mainly to one of the four presented key points:

- *Key point 1*. Students' ability to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present, is always required for carrying out proportional reasoning, however is mainly tested by items 1, 5, 6, 7 and 11 (see Appendix).
- *Key point 2*. Students' ability to draw connections to the algebraic rules for fractions when working on part/part ratios, part/whole fractions and proportions, a:b = c:d, is mainly tested by items 2, 4, 12 and 16.
- *Key point 3*. Students' ability to recognize and use a range of representations for proportions, e.g., tables, graphs, formulas, and pictures is mainly tested by items 3, 9 and 15.
- *Key point 4*. Students' ability to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity is mainly tested by items 8, 10, 13 and 14.

Several errors on items referring to the same key point indicate a lack of knowledge that should be investigated further in the follow-up clinical interview. The test items are also adapted to mirror the progression throughout the basic course. Items 1 and 4 refer to content taught in part two of the elementary course. Items 2, 3, 6 and 10 deal with content from part three, and part four is reflected in items 7, 8, 9 and 11-16 (see table 1).

Table 1. Items in relation to key points and progression from part 2 to 4 in the elementary course.

	Distinguish additive from multiplicative reasoning	Draw connections to the algebraic rules for fractions	Recognize and use a range of concrete representations	Acknowledge the properties of geometrical objects
Course part two	Item 1	Item 4		
Course part three	Item 5 Item 6	Item 2	Item 3	Item 10
Course part four	Item 7 Item 11	Item 12 Item 16	Item 9 Item 15	Item 8 Item 13 Item 14

The sources for the test items are: Gläser and Riegler (2015); Fernadéz et al. (2012); Niss and Jankvist (2013a; 2013b) and Hilton, Hilton, Dole, and Goos (2013). The items from Hilton et al. were already designed as two tier multiple-choice test items i.e. students should first decide whether an assertion is true or false then choose one statement out of four that supports the student's reasoning. The other items were adapted from their original design to a multiple-choice design, using incorrect answer alternatives either reported in the original studies or answer alternatives recalled from my experience from teaching.

The test design was tried out in two steps. First, a pilot version of the test consisting of 22 items was tried out in April 2016, by 12 voluntarily participating students with ongoing mathematics courses. This step was performed to get information about the clarity of the item's description and how long the test takes to complete. Feedback from the participants provided the insights that the test was a bit too long and that some of the items were difficult to interpret. In the second step, a revised version consisting of 16 proportional reasoning items was distributed to colleagues of mine during a teacher gathering in May 2016. Colleagues and I offered our ongoing mathematics students' to take the test voluntarily. A total of 32 students chose to participate. This aimed to check that the test discriminated among students (see results section). It was not followed up with student interviews. This step was important since a test that gives similar results among the vast majority of students would fail to give information on students' individual prior knowledge. The test took between 20 to 40 minutes to complete, without any time pressure.

A test can only produce limited information of cognitive processes (Ginsburg, 2009). Therefore, It is of great importance to follow-up the test with a clinical interview, where the students' reasoning can be confirmed, following the proposed idea in the answer alternative chosen, or adjusted if there is a misunderstanding of the task (Ginsburg, 1981). This is an important step since many students do not have Swedish as their mother tongue, which of course may cloud their interpretation of the items. Many of the students also have concentration difficulties, so a written test alone may not give a satisfactory picture of students' prior knowledge. In the interview the students get opportunity to orally communicate their reasoning to confirm or refute the found prior knowledge and potential lack of knowledge, which is a powerful method for learning about students' underlying cognitive mathematical competence (Ginsburg, 2009). The interviews are semi-structured and adjusted to the situation. The starting point is that the student should account for all his or her answers, both correct and incorrect. However, if there are a lot of errors the interview is adjusted to a sample of the items e.g. if incorrect answers is given for all part 4 items and a lot of part 3 items, the interview can be focused around the correct answers and the part 3 incorrect answers. For such a case the incorrect part 4 answers can be postponed until later on in the course. The interviews are documented with notes during the interview and afterwards, the rewritten notes are documented in the student's folder. The test together with the follow-up clinical interview gives information that provides an access point for individualized instruction, as to be shown by three cases in the results section below.

Results

Test results from the 32 participants in step two in the test design, spread between 0 to 13 correct answers, showing that the 16 items discriminate well among students. None of the students came up with the exact answering profile as another student. Two students scored zero at the test, but with different incorrect reasoning on most items. However, information from teachers gave reason to believe that the zero results may be a consequence of not having Swedish as mother tongue rather than poor prior knowledge solely. Furthermore, the analysis showed that some items caused incorrect answers from most students, i.e. 9, 11 and 14, (see appendix). The line-item (9) was solved by only three students, the dice-item (14) by two students. Also the steep hill-item (11) caused a lot of errors, solved by only four students. I will comment on these items in the discussion.

Table 2. Test results from 32 students

	Incorrect answers	Correct answers
Running laps	50% (16)	50% (16)
Number line	62.5% (20)	37.5% (12)
Riding home with Jens	50% (16)	50% (16)
End-of-term activities	57% (18)	44% (14)
Washing days	69% (22)	31% (10)
Orchestra	50% (16)	50% (16)
Loading boxes	57% (18)	44% (14)
Drawing insects	50% (16)	50% (16)
Line	91% (29)	9% (3)
Circle	66% (21)	34% (11)
Olive oil	78% (25)	22% (7)
Steep hill	87.5% (28)	12.5% (4)
Pizza	62.5% (20)	37.5% (12)
Dice	94% (30)	6% (2)
Cylinders	69% (22)	31% (10)
Tractor	66% (21)	34% (11)

Addressing the students' prior knowledge on proportional reasoning, in relation to the given theoretical underpinnings, three cases will be displayed. These cases show how the test and the follow-up clinical interview with the students provided information that provided an assess point for individualized instruction.

Case 1: Bill

It had past 16 years since Bill studied mathematics the last time. Therefore, there was reason to believe that Bill had forgotten a lot of the school mathematics as well as gained some "life experience mathematics". The test showed that Bill reasoned correctly on 10 items and incorrectly on 6. No errors were caused by test items for key point (1) Students' ability to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present. Bill's errors clustered in key point (4) Students' ability to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity, where he gave incorrect answers on all four items testing the key point. The other two errors were given for items 9 and 12. In the follow-up clinical interview we discussed the items where incorrect answers were given, to elicit the students reasoning. It became clear that Bill, in line with the given answers in the test, had difficulties with scaling in two- and three dimensions. Also, the algebraic representation of proportional functions, with focus on the limitations for the mathematical notation $y = ax$, as well as average speed, was conceptually weak in Bill's conceptual field for proportional reasoning. For the line-item Bill claimed that all lines can be written on the form $y = ax$. When I asked him "What about a line that follows the y-axis?", he could not interpret that a line following the y-axis is not possible to write on the form $y = ax$, if a is a constant. He could not say how the constant a affected the line but he recognized the representation. For the steep-hill item (item 12 in appendix) students shall decide on the average speed for a whole walk with different average speeds. One way with 3 km/h and back with 6 km/h. Bill explained to me that he added 3 + 6 and divided by 2. So, he calculated an average of the averages, which indicates that he may have a weak conceptualization of the rate, average speed. Speed is proportional to the distance, as time is kept constant. However, item steep-hill (12) involves two average speeds, over the same distance. Therefore, to just manipulate the $s = vt$ formula, fails as a method for calculating the answer.

The result on Bill's prior knowledge indicated that the assess point could be a teaching sequences to catch up on geometrical properties and scaling in two- and three dimensions to be followed up with a teaching sentence with rates, such as speed, density and price per unit. Then Bill hopefully may be prepared to start the first upper secondary course, with extra attention on the properties of linear functions.

Case 2: Caesar

While Bill's results clustered in key point four, Caesar's incorrect answers spread almost equally among the four key points. Like Bill, it has been several years since Caesar studied mathematics in upper secondary school, without reaching a grade. He reasoned correctly on 7 items and incorrectly on 9 items. Compared to Bill, Caesar's results showed a different profile on his prior knowledge. His incorrect answers clustered on items testing mathematical content from part four in the elementary course, indicating that the course design should start with catching up part four before moving on to the first upper secondary course. The follow-up clinical interview confirmed that Caesar lacked conceptual knowledge to tackle the items testing the mathematical reasoning competency required for part four in the elementary course. This became visible when Caesar despite scaffolding with questions could not reach the correct reasoning. His conceptual field for proportional reasoning seemed to lack pivotal pieces of conceptual knowledge. For the steep-hill item, also Caesar told me that he calculated an average on the two different average speeds, incorrectly concluding that the average speed for the whole walk was 4,5 km/h. When we discussed the line-item Caesar claimed to never had seen that representation before. He had no idea on how to approach the item. During the interview Caesar told me that he had not followed his class during the mathematics lessons in lower secondary school. Instead he was instructed in a small group, that to his understanding did not encounter the same challenges as his peers in the regular mathematics classroom. Of course, also this information was found most valuable for the individual design of Caesar's mathematics course. A reasonable access point for Caesar would be to start on part four, and then move on to the first upper secondary course.

Case 3: David

David was young and had recently left upper secondary school to start serving his prison sentence. His results on the test were weak, showing possible lack of knowledge in all four key points for the development of proportional reasoning. He gave correct claims on 5 of the items; where two items concerned mathematics for part two, and two items for part three. Surprisingly he also gave the correct claim on one the items that had shown to cause incorrect answers from the vast majority of students taking the test, the pizza-item (13). In the follow-up clinical interview it turned out that he had picked the answer by chance and had no idea how to approach the item mathematically. There is a probability of 0.125 to pick the correct answer by chance i.e. chose the correct true or false claim and thereafter the right reasoning alternative. In the follow-up clinical interview, David told me that he had been absent most of the time in both lower- and upper secondary school and that he perceived mathematics to be the most difficult school subject. Still, he was motivated to take up mathematics, though he believed that mathematical competencies are of great importance for life in society. Although David was a native Swedish speaker he had trouble to interpret the items. He told me that he in general is a fairly good reader, but whenever a mathematical word appeared in the text he got confused. Also David used the incorrect average on an average reasoning on the steep hill item. When discussing the line-item, David told me that he felt very uncomfortable with characters in mathematical tasks. He just picked an answer without trying to interpret the item. The test combined with the follow-up clinical interview showed that his proportional reasoning competencies partly covered part three. This raised an intricate dilemma. Shall David start on part four with some extra attention for lost mathematical content in part three or shall he start with part three, although he already seem to manage some of the mathematical content taught in part three. In dialogue we agreed to start with part three, because David preferred a smooth start on his studies.

Discussion

The aim of this study was to investigate if it is possible to design a research-based test that combined with a follow-up clinical interview gives information on students' prior knowledge on proportional reasoning so that it discriminates among students, and provides teachers with an access point for designing individualized instruction.

The test showed that it discriminates well among students both in relation to the development of proportional reasoning throughout school years, and in relation to the four key points for developing proportional reasoning, i.e. be able to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present; be able to draw connections to the algebraic rules for fractions when working with part/part ratios, part/whole fractions and proportions, $a:b = c:d$; recognize and use a range of concrete representations for proportions, e.g., tables, graphs, formulas and pictures, and acknowledge the properties of geometrical objects in two and three dimensions for calculation of scaling and similarity. The results from the second step in the design of the test distributed the students between 0 and 13 correct answers out of 16. The student cases presented above showed that the test, together with the follow-up clinical interview, provided valuable information on the students' prior knowledge, so that it could serve as an access point for designing one to one individual instruction.

The conceptual knowledge of the algebraic representation of a proportionality, average speed and scaling in three dimensions appeared to be generally weak among the participants. The line-item (8) calls for interpreting the algebraic representation of a proportionality and reason about the boundaries for the constant a, in $y = ax$. Only three students solved it correctly. This may indicate that use of different representations needs extra attention in general. Also the steep hill-item (12) caused major trouble, solved only by four students. This may indicate that the concept of average speed also needs some extra attention. The ability to calculate an average speed, given the distance and time does not require a reification of the concept average speed. You just apply the procedure of using a formula. But, for solving the steep hill-item students need to grasp the essence of the concept average speed, the relationship between distance and time. The reasoning on the three displayed cases were based on the same incorrect assumption that average speed for the whole walk can be calculated by adding (3 km/h + 9 km/h) / 2 = 4,5 km/h. This concept image may indicate that these students have a weak conceptualization of Average speed. Conceptual knowledge does not come easy for students. They need to be challenged with different situations to adjust their concept images (Tall & Vinner, 1981) and be given opportunities to develop their conceptual field of proportional reasoning (Vergnaud, 2009). Finally, the dice-item (14) caused severe trouble, solved by two students. However, if you have the information that there exist hurdles with scaling it is easy to help students realize the state of affairs with scaling from one- to three dimensions. Building blocks help students to model the situation. The block representation transforms the elusive abstraction of scaling in three dimensions to a concrete visible hands-on situation.

The three cases provided three different pictures of student's prior knowledge on proportional reasoning. Being the teacher of all three students I can tell that due to the results three different individualized course designs were set up. Both Bill and Caesar entered the course by a teaching sequence with model eliciting activities (Lesh & Doerr, 2003) giving them opportunities to investigate the properties of geometrical objects and the effects of scaling in one- two- and three dimensions. Then we paid some extra attention to the line item and other situations requiring a conceptual knowledge of average speed. Thereafter, Bill continued with number theory, while Caesar was given instruction on the difference between additive and multiplicative reasoning and hand-in assignments with tasks requiring him to distinguish additive from multiplicative reasoning and to recognize when a multiplicative relation is present. David on the other hand was introduced to real world problems where his solutions should be presented to me in oral communication with mathematical words to enhance David's mathematical language. Previously, despite having the ambition to individualize the teaching, it was difficult for me to know where to start. Students often got to start out with chapter one in a commercially produced textbook. The information provided by the test presented here gave opportunities to design individualized instruction and adapt the teaching accordingly to the students. I was given the opportunity to put my teaching competency into play (Niss & Højgaard, 2011).

Not all aspects of mathematical competencies are possible to elicit in a multiple-choice test, where students do not produce any written mathematical arguments. The views on mathematical competencies presented by Niss & Højgaard (2011) include that each of the eight competencies has a passive and an active side. The test solely requires the students to follow and assess mathematical reasoning, the passive side of the reasoning competency. It says nothing about the students' competencies to devise and carry out mathematical reasoning on the active side. However, although the test focuses on the mathematical reasoning competency it also informs on students' mathematical thinking competency, problem-handling competency and modeling competency, since these competencies are intertwined and overlapping with the reasoning competency. Together these four competencies make up one out of two overall competences associated with mathematics: The ability to ask and answer questions in and with mathematics (Niss & Højgaard, 2011). The other overall competence: The ability to deal with mathematical language and tools, covers the intertwined competencies representing competency, symbol and formalism competency, communication competency and aids and tools competency. When using the test, it is important to keep in mind that the scope of it does not cover the ability to deal with mathematical language and tools, or the ability to ask and answer questions in and with mathematics. These competences were taken up elsewhere within the course design.

While this study took place in the prison education program, this approach to test and interview students so that they can be provided with individual instruction that departures from their prior knowledge may be useful in other educational contexts for adult mathematics education.

Notes

[1] This paper concerns imprisoned students in Sweden, of which a majority are men (95%); this is why I will use the personal pronouns he and his throughout the paper.

[2] By individualized instruction I refer to an individual course design in line with both the students' prerequisites as well as the national curriculum. Teaching and learning still take place in a social setting where mathematics students communicate to justify their reasoning on modeling problems.

[3] Problem solving is also core content, as well as a goal, but the national curricula do not specify what mathematical ideas to deal with in problem solving. It is left to the teacher to decide what problems from ordinary-, society-, school- and work-life to encounter.

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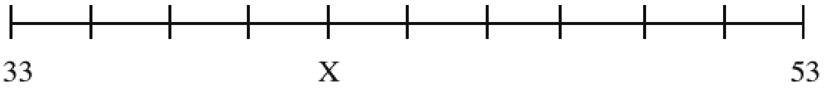
Appendix: Test items

1. **Running laps**: Sara and Johan were running equally fast around a track. Johan started first. When Johan had run 4 laps, Sara had run 2 laps. When Sara had completed 6 laps, Johan had run 12 laps.

True or **False** because (choose the best reason):

a) The further they run, the further Johan will get ahead Sara.
b) Johan is always 2 laps ahead of Sara.
c) Johan completes double the laps of Sara.
d) Sara has run 3 lots of 2 laps to make a total of 6 laps, so Johan must have run 3 lots of 4 laps to make a total of 12 laps.

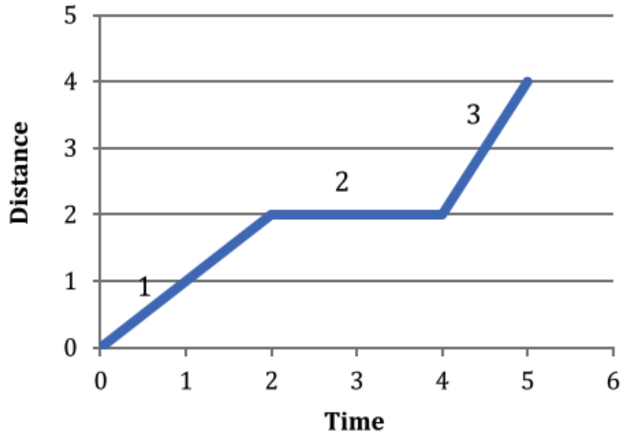
2. **Number line**: ***On this number line, X represents 37***.



True or **False** because (choose the best reason):

a) X is 4 cm along the line and 33+4=37.
b) X needs to be closer to 33 than 53.
c) X is nearly halfway so it looks OK.
d) X is 8 more than 33.

3. **Riding home with Jens**: Jens rode his bicycle home. He rode at a steady speed for a short time and then he had to rest. After his rest, he rode at double his original speed. He drew a graph to represent his journey. ***Jens' graph below is correct***.



True or **False** because (choose the best reason):

a) The distance covered in Part 3 is greater than the distance in Part 1.
b) Part 3 is twice as steep as Part 1.
c) Part 3 needs to be twice as long.
d) The times for parts 1 and 3 are the same.

4. **End-of-term activities**: This table shows the end-of-term activities voted by Year 5 and Year 6 students. ***Going to the beach is a relatively more popular choice with the Year 6 students than the Year 5 students***.

Year level	Students who chose the beach	Students who chose the movie	Total students
Year 5	8	14	22
Year 6	7	6	13

True or **False** because (choose the best reason):

a) More students in year 5 chose the beach.
b) Only 6 students in Year 6 chose not to go to the beach.
c) Fewer students in Year 6 chose the beach but there are fewer students in the class.
d) More than half of the students in Year 6 students chose the beach and less than half of the Year 5 students chose the beach.

5. **Washing days**: ***Washing powder A is the best value***.

True or **False** because (choose the best reason):

a) Washing powder A cost the least.
b) Washing powder B costs a little bit more but you get 10 more loads of washing.
c) The cost per load of washing is less.
d) Both washing powders are the same value.

6. **Orchestra**: An orchestra of 15 musicians takes 2 hours to play a concert. ***An orchestra of 30 musicians takes 4 hours to play the same concert***.

True or **False** because (choose the best reason):

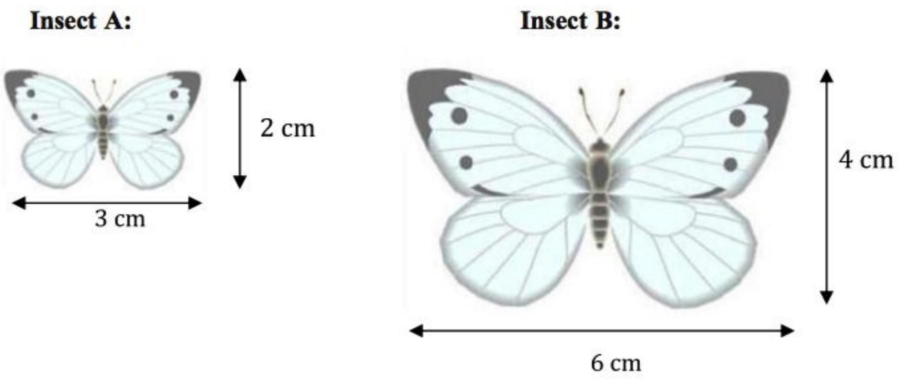
a) Doubling the number of musicians would double the time to play the concert.
b) Double the musicians should halve the time to play the concert.
c) The number of musicians does not affect the time to play the concert.
d) Adding more musicians increases the time to play the concert.

7. **Loading boxes**: Petra and Tina are loading boxes in a truck. They started together but Tina loads faster. When Petra has loaded 40 boxes, Tina has loaded 160 boxes. ***When Petra has loaded 80 boxes, Tina has loaded 200 boxes***.

True or **False** because (choose the best reason):

a) Tina will always be 120 boxes ahead of Petra.
b) Petra loads faster than Tina.
c) Tina loads 4 times faster than Petra.
d) Tina loads with double speed.

8. **Insects**: Bob has drawn two diagrams. ***The area of insect B is twice that of insect A***.



True or **False** because (choose the best reason):

a) The area of insect A is 4 times greater.
b) Insect A is half the width of insect B.
c) Insect B is twice as long as insect A.
d) Bob has only doubled one dimension.

9. **Line**: We know that an equation on the form *y*=*ax* (*a* is a constant) gives a line through (0,0) in a coordinate system. **Every line through (0,0) has an equation on the form *y*=*ax*, (*a* is a constant)**.

True or **False** because (choose the best reason):

a) There exist a line through (0,0) that cannot be written on the form *y*=*ax*.
b) The reasoning goes both ways.
c) All lines can be written on the form *y*=*ax*.
d) You cannot know without investigating an infinite amount of lines.

10. **Circle**: **Simon says that if you draw a new circle with half the diameter of another circle, the new circle will have half the perimeter and half the area of the other circle**.

True or **False** because (choose the best reason):

a) If the diameter is halved, the perimeter and area is halved.
b) The area will be ¼ and the perimeter ½ of the original.
c) You cannot know without knowing the length of the diameter in the new circle.
d) You cannot know without knowing the length of the diameter in the original circle.

11. **Olive oil**: You'll make a salad dressing according to this recipe for 100 milliliters (ml) dressing. **You need 110 ml olive oil to 150ml dressing**.

Olive oil	60 ml
Vinegar	30 ml
Soya sauce	10 ml

True or **False** because (choose the best reason):

a) It is far too much oil.
b) You will need 1.5 times oil.
c) Since the vinegar and soya sauce is 40 ml, the rest will be oil.
d) You need 100 ml oil.

12. **Steep hill**: There is a path up a quite steep hill in Athens. Rickard, who is in good shape, is going up the hill in an average speed of 3 km per hour. He goes down in double speed. **Richard's average speed for the whole walk is 4 km per hour**.

True or **False** because (choose the best reason):

a) The speed is higher than 4 km per hour.
b) The speed is 4 km per hour.
c) The speed is 5 km per hour.
d) You cannot know if you don't know the distance.

13. **Pizza**: A pizzeria serves round pizzas of various sizes. The smaller pizza has a diameter of 30 cm and costs 30 SEK. The larger has a diameter of 40 cm and costs 40 SEK. **The large pizza gives more pizza for the money**.

True or **False** because (choose the best reason):

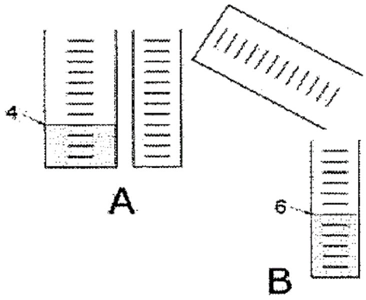
a) The small pizza gives more pizza for the money.
b) It is the same cost, 10 SEK per diameter pizza.
c) The small pizza is more expensive per area unit.
d) You cannot decide if you cannot see the pizzas.

14. **Dice**: A wooden dice where all edges is 2 cm weighs 4.8 g. **A wooden dice where all edges is 4 cm weight 19.2 g**.

True or **False** because (choose the best reason):

a) The weight increases 4 times if the edge doubles.
b) The weight increases 6 times if the edge doubles.
c) The weight increases 8 times if the edge doubles.
d) The weight doubles if the edge doubles.

15. **Cylinders**: Below are drawings of a wide and a narrow cylinder. The cylinders have equally spaced marks on them. Water is poured into the wide cylinder up to the 4th mark (see A). This water rises to the 6th mark when poured into the narrow cylinder (see B). Both cylinders are emptied (not shown) and water is poured into the wide cylinder up to the 6th mark. **The water would rise up to the 8th mark if it were poured into the empty narrow cylinder**.



True or **False** because (choose the best reason):

a) The answer cannot be determined with the information given.
b) It went up 2 more before so it will go up 2 more again.
c) It goes up 3 in the narrow for very 2 in the wide.
d) The second cylinder is narrower.

16. **Tractor**: When the big wheel of a tractor turns 5 times the small wheel turns 8 times. **When the big wheel turned 7 times the small wheel have turned 11 times**.

True or **False** because (choose the best reason):

a) The small wheel makes 3 more turns than the large one.
b) One has to find out the answer by observation.
c) The small wheel turns about one and a half time as often as the big one.
d) Both wheels will have turned two extra turns.

Jain, P. & Rogers, M. (2019). Numeracy as critical thinking. *Adults Learning Mathematics: An International Journal*, 14(1), 23-33.

Numeracy as critical thinking

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Abstract

Adult learners preparing for university level study have varying degrees of prior experience, confidence, ability, motivation and need when it comes to mathematical tasks. The development of numeracy skills is a core aspect of university preparation, with many university courses requiring a certain level of mathematical literacy.

The standard model of numeracy support in enabling programs essentially replicates what students learned (or failed to learn) in high school. For adult learners with previous negative experiences, this approach can be intimidating, while others perceive high school mathematics as irrelevant and unnecessary to their chosen field of study. Adult learners require an entirely new approach to numeracy support.

Our approach has been to turn the standard model on its head, by recognising numeracy as a form of critical thinking that requires much more than an ability to manipulate numbers. The University of Tasmania Academic Numeracy unit utilises Watson's (2009) Four Resource Model of Numerical Literacy to focus on critical evaluation of information, in both everyday and academic situations. The development of higher-order critical thinking skills forms the core of our unit, with explicit mathematical instruction provided in service of situating and critiquing real research. This approach has allowed us to develop critical numeracy skills in a way that is relevant, meaningful and valuable to all learners, whatever their future academic endeavours.

Keywords: numeracy, critical thinking

Introduction

The importance of numeracy skills in the workforce and daily life is becoming an increasingly pertinent issue. The relevance of mathematics is receiving attention both in Australia (Marginson, Tytler, Freeman and Roberts, 2013; Office of the Chief Scientist, 2014) and worldwide (Breiner et al., 2012) as STEM (Science, Technology, Engineering and Mathematics) education is seen as essential to support the modern knowledge based economy (Abbott-Chapman, 2011). As such, there is an expectation that university graduates will be both literate and numerate, with numeracy skills being increasingly recognised as a key outcome of university study. Deputy Vice-Chancellor of the University of Technology Sydney Shirley Alexander says, "We think that to produce graduates that are really going to contribute to society every single graduate needs to have numeracy" (Alexander, 2016).

There is a growing recognition that numeracy skills form an important precursor to successful study at the university level. Concerns about school leavers' level of mathematics preparedness for entering STEM disciplines is garnering high levels of focus. Sydney University has recently reversed a decision to waive mathematics pre-requisites, and from 2019 will reinstate 2-unit mathematics as a prerequisite for entry into 62 degrees (Alexander, 2016)., including economics, commerce, engineering and IT, psychology, pharmacy, veterinary science and science. These courses have an indisputable mathematical component. However, an understanding and ability to think critically about numerical concepts and information is important for students in all academic areas (Gazit 2012). Indeed Australian Prime Minister Malcolm Turnbull has recently suggested that Maths/Science should be a prerequisite for any university study. "More universities are requiring and they should all require in due course that mathematics or science should be a prerequisite school subject to have completed to go onto university." (Bagshaw, 2016).

The University of Tasmania Tertiary Numeracy Enquiry (Skalicky et al., 2010) audited the level and type of mathematics and numeracy in undergraduate courses across the university and the level of numeracy needed to successfully complete an undergraduate course. It was conducted in response to staff concerns about a mismatch between numeracy requirements of University courses and students' ability to meet these demands. Findings emphasise the need for numeracy skills across all disciplines, with nearly every School reporting that data representation, analysis, and interpretation were specifically required for their courses. Academic staff also identified concerns about the critical thinking and problem solving abilities of students, which is consistent with findings at other higher education institutions in Australia (e.g., Chapman, 1998, Galligan & Taylor, 2008; Kemp, 1995) and internationally (e.g., Gill & O'Donoghue, 2006), as well as at the pre-tertiary level (Office of Tasmanian Assessment, Standards and Certification, 2008).

Development of the Academic Numeracy unit

The University of Tasmania Academic Numeracy unit was developed in response to the findings of the Tertiary Numeracy Enquiry, to ensure all enabling course students had a base level of numerical literacy, regardless of their future area of study. The focus of the core, semester-long Academic Numeracy unit is a critical thinking approach to build numerical literacy.

Pre-degree Programs at the University of Tasmania is the entry level course for students who have not traditionally considered tertiary education (including first-in-family, mature-aged, low socio-economic background, those not meeting entry requirements and students with health or learning issues). This course acts as an enabling, pathway program for students who wish to enter bachelor level study and equips adult learners with skills needed to successfully transition to degree-level study.

Pre-degree Programs adult learners have different and diverse mathematical backgrounds and abilities, and varying levels of confidence and motivation. Many are intending to undertake non-STEM degrees (Education, Nursing, Arts, and Social Sciences etc.) and may not see the need to study mathematics. The student cohort is non-traditional, 27% aged over 30 years, 24% last studied mathematics more than 10 years ago, and 30% have not studied mathematics beyond Grade 9/10. Approximately 13% of the students are from culturally and linguistically diverse backgrounds. Many students have disengaged from study (particularly mathematics) earlier than year 10. In Tasmania, overall tertiary participation of 14.3% is well below the national average of 18.8%, with some regions, such as the Cradle Coast being well below average at 8.7% (Australian Bureau of Statistics, 2011). Many students enrolling in Pre-degree mathematics programs have either not attempted or not done well in college mathematics for a variety of reasons, including lack of ability, interest, motivation or perceived relevance, ability, unsuitable teaching techniques and prior negative experiences.

Background - what we used to do and why we needed to change

Previous entry-level mathematics units in the University of Tasmania Pre-degree Program were essentially a replication of high school mathematics with traditional topics (number, algebra, geometry, and statistics) delivered in traditional ways. However, the traditional content and delivery was not producing good results: students lacked confidence, their liking of and ability in mathematics was not increasing and many did not see its relevance to non-STEM areas of study. Many students objected to the fact that the unit replicated high school mathematics and they felt that they had been there, done that!

The challenge was to design a curriculum that was significantly different to the standard unit. We needed to meet a number of requirements, in a way that resolved multiple, somewhat competing, constraints. The new unit would:

- Prepare maths-inclined students for further mathematics study
- Prepare nursing/paramedic practice/education pathways students for mathematics entry requirements
- Prepare students entering non-mathematics disciplines for general academic success in first year units
- Be unlike high school mathematics courses
- Achieve all of the above while catering to a non-traditional student cohort with vastly diverse mathematical backgrounds and abilities.

Additionally, we were keen to improve our student's attitudes towards mathematics by building our student's confidence in their mathematical abilities and fostering in them an understanding and appreciation of the relevance of mathematics in their lives.

The critical thinking approach

In designing Academic Numeracy, we were well aware of a larger responsibility to our students: for many of our students, this may be their last formal mathematics instruction. What should a "last chance" mathematics unit focus on? How much mathematics is "enough" (for academia, for life)?

High school mathematics typically divides mathematics into specific topics: number and algebra, measurement and geometry, statistics and probability (Australian Curriculum, Assessment and Reporting Authority, 2016). Rather than mirror these topics, we focus on developing skills that go far beyond being able to do calculations and solve word problems. We realised that mathematical ability is in fact a form of literacy, in which critical thinking performs a crucial role. Numerical literacy involves making sense of information that contains mathematical concepts and expressions, making connections with other ideas and contexts, and creating meaning and knowledge. In short, it involves being able to think critically about numerical information. Critical thinking, then, is the both the foundation and the goal for the Academic Numeracy unit. The unit aligns with Jane Watson's Four Resource Model for critical numeracy (Watson, 2009), which in turn is based on a framework for critical literacy (Freebody & Luke, 1990). The Four Resource Model (Figure 1) highlights the components of "de-coding, meaning-making, using and analysing" within the critical thinking framework. The application of this framework in the Academic Numeracy unit involves engaging our students in understanding, analysing, evaluating and making decisions about everyday and academic issues that involve mathematical concepts.

The critical thinking approach makes the unit very different to previous high school courses our students may have encountered. With the development of higher-order critical thinking skills at the core of the unit, the unit is immediately relevant and valuable to all of our students, whatever their chosen field of study.

To meet entry requirements for STEM and other pathways, we recognised the need to provide explicit mathematical instruction within the unit. However, the diversity of our student cohort discouraged a traditional approach. One consideration that was paramount in our unit design was the desire to prepare students for further study by developing not only critical numeracy skills, but also an understanding of mathematics in academic disciplines and the world. As statistical research, data collection and analysis forms the heart of most of the research people will encounter in their day-to-day lives and future academic studies, we chose to make statistical literacy the focus of the unit. As the economist, Max Roser, who runs the Our World in Data project (https://ourworldindata.org/), notes:

> The kind of maths that people are taught at school focuses on algebra and calculus, which they hardly ever use later in life. … You use statistics all the time – for the weather forecast or calculating your income. And whether you're talking about it with other academics or in the pub, these are topics that matter to people.
>
> (Roser, quoted in Cumming 2015)

De-coding Terminology Maths ideas and key concepts Different ways numbers are used and represented Key mathematical processes and procedures	**Meaning-making** What do I know? What is the text about How do mathematical concepts make sense and help me understand this context? What is confusing or misleading? Are there other possible meanings?
Using How are the numbers or mathematical concepts significant of useful? What is the purpose of the text How might the text be used to promote different viewpoints? What are the possible applications? What are the likely impacts? How would I use this text and what decisions would I make based on it? Am I now thinking about issues and mathematical concepts differently?	**Analysing** Is it true? What is the evidence? Are mathematical concepts used appropriately? Is it logical and consistent? Is it researched appropriately? From a reputable source? Is it fair? Different views, values, perspectives or types of research. What is missing? How does it position me?

Figure 1. Watson's Four Resource Model for Critical Numeracy (2009)

With this in mind, the Academic Numeracy unit centres around critical thinking about statistical research, using both everyday and academic examples, with explicit mathematics instruction provided as needed, to clarify, situate, and critique actual research. In this way, students develop the mathematical skills needed for entry requirements, alongside higher-order critical thinking skills that are of value to all students regardless of their future academic endeavours.

How we do it? The structure and focus of our unit

Forefront in the minds of staff in the Academic Numeracy unit is the nature of the cohort of students. The Australian Association of Mathematics Teachers (AAMT, 1998) discusses the importance of a grounded appreciation of context and this is particularly important for our adult student cohort who, as noted above, may have disengaged from high-school mathematics, have poor mathematics' attitudes and do not see the relevance of algebra and trigonometry to their lives. Kemp and Kissane (1990) consider that "If students have not learned how to use mathematics across the curriculum unless explicitly asked to do so, it is unlikely that they will use mathematics beyond the mathematics classroom" (p. 117). Additionally, Maciejewski (2012) and Clute (1984) found that students with high math anxiety benefited from an approach based on explicit instruction. To address this issue, the Academic Numeracy unit provides sequenced skill-based, computational exercises aligned with in-context use of those mathematical skills to make sense of everyday and discipline specific research. The focus is to engage students in using mathematical tools to critically evaluate and make meaning of mathematical information using critical thinking skills. Students are encouraged to consider mathematics as a way of thinking and communicating with direct relevance to their lives, future study and career aspirations. Further details are provided in the sections below.

The unit structure employs principles of constructive alignment (Biggs & Tang, 2007), with components structured around Watson's Four Resource Model for critical numeracy (Watson, 2009). Opportunities for students to engage in self-directed study develops their understanding and skills from different entry points. Additionally there are mechanisms for self-checking their understanding and formal assessment processes that engage with both numerical literacy and critical thinking.

Integral to the sequential development of skill and concept is the linkage between different aspects of the unit. Each component (lectures, tutorials, open learning resources and assessment tasks) focuses on aspects of Watson's Four Resource Model (2009) and provides recursive and iterative opportunities for students to develop and check on construction of knowledge, skill level development and the ability to apply number skills to critical thinking processes. Within this framework, opportunities are provided for students to revisit earlier tasks, and pursue additional work at their own level and pace (Figure 2).

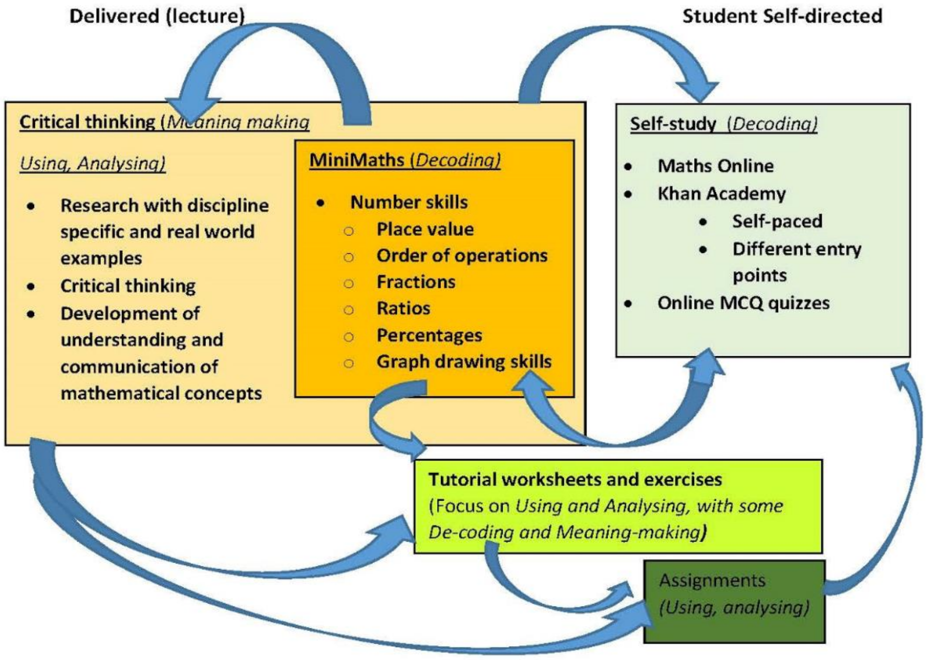

Figure 2. Relationship between components of Academic Numeracy highlighting the recursive and iterative nature of opportunities for students to develop skills.

Lectures

Lectures utilise all four parts of Watson's (2009) Resource Model for critical numeracy with content based around statistical literacy. Lectures provide opportunity to develop understanding of terminology and number skills (de-coding), while placing mathematical concepts within a meaningful and relevant context (meaning-making). Lectures analyse how numbers and mathematical concepts promote particular viewpoints and allow predictions and decisions to be made (using). Students are encouraged to examine evidence, assess assumptions and perspective, and evaluate and reflect on their own position (analysing). Throughout the lectures, students work through problems, engage in individual and group activities, self-check understanding and clarify and develop concepts and skills with peers and staff.

The statistical focus of the unit is not typical - statistical calculations form only a small part of the broader notion of statistical literacy. We aim to develop informed citizens who recognise the value of statistics and large-scale population research in providing insights into current issues, where content and delivery emphasises the importance of data in telling a convincing story about the world. The use of popular media as a source of text and data engages students in the four domains of Watson's Four Resource Model (2009). Graphs, tables, case studies and evidence presented in legal cases is used to engage students in numerical concepts relevant to everyday life. This is where the mathematics is in everyday life! We try to bring in topics that matter to people, and the research around these topics that is based on statistics (not algebra and calculus).

Example 1: Climate Change.

To introduce the topic, students consider data from current and recent newspapers highlighting heatwaves, droughts, floods etc. Long-term data in the form of reports, tables and graphs from the Australian Bureau of Meteorology and other reputable sources builds the story. Students then examine their own perception of climate change, with discussion regarding what needs to be known to make informed decisions. Students are challenged to undertake meaning-making and use data to critically evaluate and analyse the concept of climate change. Previous skill development in percentages and ratios are used in context to support understanding and interpretation of the statistics and graphs.

Example 2: Mandatory bike helmet legislation.

A mandatory bike helmet law lecture provides an example of integration of critical thinking skills and mathematical skills developed in the unit within the context of Watson's Four Resource model (2009). Initially the lecture involves students engaging in Watson's concepts of meaning-making, using and analyzing, by examining their personal views and considering the wider context of mandatory helmet laws. Discussion centers around interpreting research and popular articles that promote differing view-points regarding helmet laws. Class discussion culminates in developing ideas about what types of data would be required to assess the effect of mandatory helmet laws.

Additional web-based research into the effectiveness of bike helmets leads to examination of an original research article by Dinh et al. (2013). The research results are discussed and interpreted using statistical concepts that have been developed within the unit (percentages, ranges, demographic data, sample selection, correlations and causation etc.) This involves the process of integrating de-coding with using, meaning-making and analyzing to integrate mathematical skills in an example of a broader context.

MiniMaths

Within the formal lectures, "MiniMaths" comprises a short numeracy skills focussed component (part of Watson's (2009) de-coding) to introduce or reinforce numerical concepts required to engage in critical thinking. This explicit mathematics instruction supports students and prepares for further study in the other critical numeracy domains. Varying backgrounds made it essential to include this component, while delivery also needed to be sensitive to students already skilled in basic numeracy concepts. The focus of the unit is on developing numerical critical thinking and the inclusion of the MiniMaths component is innovative as it is provided for students to develop mathematical skills at their own pace, at their own level.

Students are presented with a concept (such as order of operations, rounding, directed numbers, decimals, drawing graphs) in a short focused mini-lecture, followed by an opportunity to explore and self-assess their understanding. Additional self-study opportunities are provided, including relevant self-paced open learning resources (Khan Academy clips and guided lessons through the web-based program MathsOnline).

Students can work through suggested MathsOnline lessons to develop skills introduced in MiniMaths. Students with lower level entry skills can search for and work through lessons appropriate to their individual level at their own pace in privacy. Students with advanced level skills can extend and expand their knowledge with access to higher-level lessons and questions. Through a process of self-assessment of their skill and confidence in the MiniMaths lesson, students can do as much or as little as they individually need.

Tutorials

Tutorials engage students in Watson's (2009) domains of using, meaning-making by examining texts and research articles to identify mathematical concepts and make sense of the material in context, utilising the terminology and mathematics concepts covered in the lectures and "MiniMaths" components. For example, students are asked to design and conduct a short research study to identify average signature length. As a tutorial group, students develop the research question, design an experiment, collect and analyse the data by performing basic analysis of central tendency, drawing tables and graphs to identify distribution of data to draw conclusions. In other tutorials, students are involved in Watsons (2009) domain of analysing to assess whether mathematical concepts are being used appropriately by examining the evidence provided in texts and to understand their own position.

Each weekly tutorial provides the support that links with the MiniMaths, weekly quiz, tutorial exercise and numerical component that is introduced in the statistical research component of the unit.

Weekly online quizzes

Students engage in self-checking their skill development with low weighted weekly online quizzes related to the MiniMaths topic. The quizzes are focused on Watson's (2009) domain of decoding to provide development of underpinning numeracy skills which are then applied in critical thinking contexts. Additionally, the MiniMaths and quiz weekly topic forms part of the following week's tutorial which enables tutors to check on skill development and provide students with specific and focussed support.

Open Learning Resources (OLRs)

To support student development, open learning resources are used to provide self-paced learning support for students. These OLRs provide opportunities to engage students in all domains of Watson's (2009) Four Resource model by exposing students to examples of data representation and research studies that highlight skills or extension topics covered in the lectures.

But where's the maths?

The choice of topics within the unit are selected by the current and future needs of the students to understand and interpret research results within their non-STEM degree context. The focus is on conceptual development but with some explicit instruction in

- Statistical literacy and data analysis - research design, variables, graphs (drawing and interpretation), tables, calculations of central tendency, normal and skewed distribution
- Proportional reasoning (as research results are often expressed using percentages and ratios)
- Algebra (as a way of formalising patterns and linked to relationships between variables)
- Finance (for personal numeracy)

Statistical literacy is a core component and is the focus of the first six weeks, as well as the overarching theme. Although there are some calculations of measures of central tendency, this is done only within the context of evaluating statistical research, with the emphasis on critical thinking about what the research results are telling us. The development of number facts is integrated with statistical literacy by discussing aspects such as sample size and figures as raw numbers and as percentages or ratios. The focus is on understanding the figures in context to interpret and find the significance of the results.

Student perspective

Many students report that they find the unit surprising and challenging. The focus of the unit is very different to what they were expecting or their previous experiences with mathematics, resulting in changes to their perceptions.

> Overall my view and attitude towards learning Maths have dramatically changed in direct response to what I have learned this semester from the unit.

Part of our concerns about the original content and structure of the previous Academic Numeracy unit was that it mimicked high school mathematics and our aim for the new unit was to challenge students to think differently and apply mathematics in a context to which they had not previously not been exposed. For some students the challenge is unexpected. Based on previous study of traditional mathematics topics students expect that they will be able to easily complete the assessment tasks. However, the critical thinking focus and the requirement for meaning-making, using and analysing produces a more demanding unit.

> I found that I had to work in order to comprehend the concepts.
> The Assignments were challenging but very enjoyable to partake in.

Students respond positively and develop confidence with mathematics, with many seeing the value in critical thinking about numerical information and the importance of developing skills in the decoding domain to support higher order thinking.

> Learning how to critically evaluate information with a mathematical view. Given that I was a less than successful student in high school many years ago, I have large gaps in my mathematical knowledge. This unit has helped to fill these gaps with things such as percentages, fractions. The ability to utilise mathsonline as an additional source of education which is tied in with the weekly lessons and helps clarify and reinforce what has been taught is also of great benefit.

Particularly relevant for the cohort of students is the increase in confidence and reduction in maths anxiety. Many students report that they are less fearful of mathematics.

> At the beginning, I was very nervous about taking this unit but now I feel much more confident in mathematics than I could ever imagine.
> I had really struggled with my maths at high school and developed a negative relationship and I was extremely nervous about starting this course. I can now quite comfortably say that I really enjoyed this course, and was eager to see my results… I am currently looking into trying the next level up.

Students find that they transfer the skills learned in the classroom to their everyday life, talk to their families about the research studies and learn to see the relevance of mathematics to their career aspirations.

> I found myself challenging myself when I was offered an opportunity to investigate numerically in day to day life, previously I would have let it go.
> I am pleased that I am able to use the things we learnt in real life as well, such as calculating percentages, thinking critically about information presented by media sources, and also tax.

The unit requires students to read research articles and write responses explaining their answers using supporting evidence from the research and/or graphs. Ackland (2014) argues that literacy may be an additional barrier. This may be true if the assessment tasks are primarily a procedural mathematics problem dressed up (e.g. a word problem). However, we design assessment to be primarily a higher order conceptual task (with perhaps some procedural mathematics required to fully appreciate it) and utilise current topics and issues to engage students and promote the relevance of mathematics to their lives and study. For example, assessments relate to changes in learner driver speeds, health issues (heart attack rates, dental costs, autism, and happy people getting more done at work) and current finance issues.

One of our original concerns was how to provide learning opportunities that addressed individual student entry points. Our answer was to provide the short MiniMaths lecture each week to introduce the topic, provide links to open learning resources and MathsOnline lessons and then give students an opportunity to self-check progress with weekly quizzes. This allows students to develop their own understanding from their individual base level.

> I found the MathsOnline lesson to be very helpful and a really useful resource to cement the techniques learnt in the lectures and tutorials.

The material and resources were particularly interesting and helpful in assisting me to grasp various numeracy concepts. The additional online mathematics aids were excellent and being able to access so much information at my own pace lead to my improvement in the subject.

The University of Tasmania student feedback system (eVALUate) provides quantitative and qualitative responses from students. Students from 2015 and 2016 indicate high levels of overall satisfaction with 96% of responding students indicating overall strong agreement or agreement in favour of the unit and learning opportunities provided. Specifically students respond positively to the following unit evaluation questions in relation to the unit.

- The learning experiences helped me achieve the learning outcomes - 96% agree or strongly agree
- The learning resources in this unit help me to achieve the learning outcomes - 96%
- The assessment tasks in this unit evaluate my achievement of the learning outcomes - 97%
- I am motivated to achieve the learning outcomes in this unit - 93%
- I make best use of the learning experiences in this unit - 92%
- I think about how I can learn more effectively in this unit - 91%

Authors' reflection

Development of the unit has been a challenging and rewarding experience. For those involved, the focus on critical thinking about numbers has been a process that has required a change in the way we think about mathematics and what it means for students. Rather than focusing on process as a goal, we have been constantly looking for relevant and interesting examples that provide an opportunity to integrate number skills. This provides students with a reason to develop number skills that promotes an understanding of the concepts that underpin the examples.

It has been challenging to align the two strands and still ensure that we meet the requirements for students who are progressing to non-STEM degrees, while providing an engaging unit that aims to alter the attitudes of many students to mathematics

Reflection of staff unit teaching staff

Staff involved in the unit have identified that the critical thinking approach in this unit is desirable for adult learners, as being able to critically analyse a piece of research or even a newspaper article for soundness will serve them well both in university as well as generally in life. The focus on critical thinking and analysis using mathematics has the effect of producing two groups of students: one group loves the approach and the other is wary and perhaps confused as to why there is not more standard mathematics in the unit. Staff find that the critical thinking approach has the welcome effect of “levelling the playing field”: Students who are comfortable with calculation-based tasks still find the critical thinking aspects challenging and rewarding, whereas students who are initially daunted by standard mathematics find that our approach allows them to engage with mathematical thinking in a way that has been previously inaccessible to them. Most of our students come to appreciate that critical thinking is the cornerstone of numerical literacy, and understand that the skills they learn in this unit will stand them in good stead throughout their future study.

Summary

The development and delivery of Academic Numeracy as a mathematics unit with a focus on critical thinking has been a challenging and rewarding process for staff that has required us to think differently about mathematics education. We have developed a unit that engages adult learners in thinking critically about mathematical information, while providing underpinning explicit mathematics instruction to enable students to appreciate and utilise mathematics across their preferred discipline area and within their lives. Responses from students have been generally positive, with some students surprised at their ability and enjoyment of mathematics. By being cognisant of the diverse entry levels of skills we have created a unit that allows self-paced development in procedural mathematics alongside higher order critical thinking skills to provide a rich and relevant learning experience for all of our students.

> This was my favourite subject…. I finally understood and ‘got’ numeracy.

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Divorce, evil, and the regime of terror: personal characterisations of mathematics in the lives of mature students

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Abstract

Increasing numbers of students attending higher education institutions in Ireland take obligatory mathematics modules – service mathematics – as part of their programmes of study. A dislike of mathematics is frequently expressed among students, particularly mature students, some of whom experience mathematics anxiety. However, despite their experiences with mathematics, mature students demonstrate motivation and resilience in respect of their engagement with mathematics. As part of a mixed methods study, the author endeavoured to explore the mathematics experiences of mature students throughout their lives, and to identify those incidents that have impacted significantly on the students' engagement with mathematics. Twenty mature students were asked to characterise their relationship with mathematics. This paper[1] presents these findings, depicting an interesting variety of responses.

Keywords: mature student, service mathematics, mathematics life story, mathematics anxiety

Introduction

Mature students[2] represent a heterogeneous cohort in undergraduate programmes in higher education (Bloomfield & Clews, 1994; Lynch, 1997). Their challenges can be manifold and can differ considerably from 'traditional students' who have completed school and progressed straight to higher

[1] The present paper stems from the author's (Ryan) Ph.D. research into the existence of mathematics anxiety among mature students; the paper also builds on the findings of a previous paper presented at ALM-23 and published in 2017 (Ryan & Fitzmaurice, 2017).

[2] In the Irish context, a mature student is defined as being 23 years or more on January 1st in the year of entry or re-entry to an approved course (www.hea.ie, 2018)

education (Brady, 1997). Returning to education as a mature student can be intimidating, as can the exposure to different teaching and learning methods compared to what mature students would have encountered at school (Hadfield & McNeil, 1994; Chinn, 2017). Service mathematics (Macbean, 2004; Gill & O'Donoghue, 2008) is frequently an obligatory module in undergraduate programmes and may pose a further challenge for the mature student, who may not have associated mathematics as a likely component of their chosen discipline of study (Fitzmaurice et al., 2014).

Some mature students experience mathematics anxiety, defined as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (Richardson & Suinn, 1972: p.551). However, mature students are highly motivated (Knowles et al., 1998), and any negative issues they have with mathematics are usually dealt with or put aside in order to enable them to engage with mathematics and facilitate progression with their programme of study (Hadfield & McNeil, 1994). In this regard, mature students tend to seek help with mathematics (Fitzmaurice et al., 2014), frequently availing of mathematics support facilities on campus. Behind the scenes, however, some mature students may have experienced a turbulent past with mathematics, that may be responsible for their approach to and level of engagement with the subject in higher education (Hembree, 1990; Tobias, 1993), but which may not be disclosed (Tobias, 1993). To this end the present study endeavoured to uncover the stories behind mature students' experiences with mathematics in order to get a sense of how they feel about the subject as experienced throughout their lives.

The students who participated in this study were asked to share their mathematics life stories (Briggs, 1994; Coben & Thumpston, 1995; Drake, 2006), which culminated in the student giving a theme to their relationship with mathematics. Affording the interviewee the opportunity to consider a personal theme gives them the chance to look back and – in choosing a theme – enhance their understanding of their life story (McAdams, 1993) with reference to the path taken as well as significant events and relationships along that journey (Atkinson, 1998). Thus, as the mathematics life story aims to shed light upon the life course that the student associates with their engagement with mathematics, the personal theme for their relationship with mathematics helps to elucidate how they feel about the subject in the context of their mathematics life story (Atkinson, 1998).

Mixed method research design

With an overarching focus on investigating the existence of mathematics anxiety among mature students studying service mathematics in Ireland, the research design comprised a sequential mixed method approach (Mertens, 2015); phase one was a survey targeting mature students who are studying service mathematics in higher education undergraduate programmes across the University and Institute of Technology[3] sectors; this captured the respondents' scores on the 23-item Mathematics Anxiety Scale U.K. (MAS-UK, Hunt et al., 2011), as well as some biographical data (N = 107). This phase was followed by semi-structured interviews with twenty mature students who opted in from phase one (Ryan & Fitzmaurice, 2017).

To get an insight into the participants' feelings about mathematics, and to elicit the stories of mature students' engagement with mathematics throughout their lives, the interview format used an

[3] An Institute of Technology is a further and higher education institution which focuses on teaching and learning, purpose-driven research and development, business support and incubation, and civic engagement and public service (www.thea.ie, May 2018).

adapted version of McAdams's (1993) Life Story Framework; this framework was deemed fit for purpose as it aims to elicit the stories from different stages of the interviewee's life course – childhood, adolescence, adulthood, future, as well as examining strategies adopted throughout their life to-date, and their overall characterisation of their life experience. This framework aptly paralleled the course of a mature student's typical engagement with mathematics throughout their lives; consequently, the format of the interviews was guided by this structure. Participants were invited to give accounts of their engagement with mathematics at primary school, and post-primary school, as well as mathematics beyond school – either in the workplace or in further engagement with education –, as well as the role of mathematics in their decision to return to education, and the part they see mathematics playing in their future. The framework also provided scope in the interview to examine the strategies the participants had taken to engage with mathematics, as well as the characterisation of mathematics in the form of a personal theme for their relationship with mathematics. It is the responses to this last part of the framework that inspired this paper, with the focus on the characterisation of mathematics by the participants, and the stories behind them in order to illuminate the themes.

Findings

The title of this paper presents three of the themes characterised by the students: divorce, evil and the regime of terror.

Divorce

The first theme – divorce – was presented by Neo, a male student with a low MAS-UK score (29).

> It was like a marriage that broke up and got back together. I took it for granted for a while when I was younger, and then we parted terms and it wasn't amicable. … I had a divorce. And we missed each other, and after a few rendezvous with other areas of my life, we got back together, [and are] looking forward to a bright and prosperous retirement together, so it's onward and upward. (Neo)

Neo dropped out of school at age 15, prior to which he recalls being frustrated with mathematics, as he had lots of questions about mathematics but could not get answers, and the difficulties with mathematics got worse. After school, he worked as a manual labourer on building sites. During this time, he was exposed to engineering decisions, but could not contribute. He returned to higher education after doing a one-year access programme. In higher education he has relied heavily and engaged positively with the mathematics support facility, resulting in a very positive experience with his programme of study in the discipline of engineering.

Evil

The second theme – a necessary evil – was presented by Ken, a male student with a low MAS-UK score (31).

> It's a necessary evil … I'll get to a certain level in maths and that will be it, and I don't think I'll ever be totally comfortable with it. It will always frustrate me, and I'll always be wary of certain aspects of it, because I don't understand it. … A necessary evil: I do it, I do what I can with it, but I'm never going to be a shining star. (Ken)

Ken enjoyed school mathematics, but from fourth year (high school, UK) there was a heavy focus on algebra without obvious relevance. He had over 30 years of a gap since leaving school and entering higher education. Sometimes he cannot follow instructions in class because he cannot understand a concept the first time it is presented. In mathematics examinations, he finds that questions can be vague, and that confuses him. He wonders if that is done deliberately by the lecturer. He focuses on sub-questions with high marks and admits that he struggles with the smaller details.

The regime of terror

The third theme – the regime of terror – was presented by Jon, a male student with a moderate MAS-UK score (59).

> I've developed a liking for numbers, I really have, but in early years my god, … [for most of primary school] I lived under a regime of terror. There was incidents within those years … if you got the slightest thing wrong, you got beat. … And I suppose it wouldn't be fair to say the maths was terrorising me, it was the system terrorised me. And maths is the catalyst, the thing that's causing me all my problems, you know. So I just avoid it. (Jon)

Jon suffered physical abuse at primary school for getting mathematics wrong. He missed six months of first year at secondary school due to illness, and never caught up with the mathematics he missed out on. He has avoided mathematics as much as possible in his life, even though his work before entering higher education involved calculations which he had no problems with. He had already started his degree programme when he realised he would have to do two mathematics modules and considered leaving the programme. However, with the help of the mathematics support centre he tackled the mathematics and succeeded in both modules.

Other themes

The interviews gathered a variety of other personal themes, ranging from positive to negative characterisations, as well as contrasting themes. Positive themes included:

- the universal significance and logic of mathematics;
- the clarity of mathematics (i.e. you get an answer);
- the Eureka moment and sense of illumination that it brings;
- a gel that binds the coursework together;
- a confidence builder.

These themes present a sense of appreciation of the importance of mathematics to the real world, its relevance and our dependence on mathematics. Mathematics is viewed as particular and clear, in that you get an answer. Its presence across modules helps students see the connection of mathematics to the real world and future careers. Proficiency in mathematics lends itself to incremental confidence. The students who presented these themes comprise four males and one female with MAS-UK scores ranging from 29 to 56.

In contrast, some of the more negative themes included:

- Fear and trying to make sense of it;
- Mathematics not liking you;
- Mathematics is not my friend;
- Inaccessibility of mathematics;
- Having to do it to get through;
- Avoid it and do not (want to) embrace it.

These themes contrast with those above and exemplify the challenges these students face in terms of how they feel about mathematics. There are feelings of fear and dislike of mathematics, that mathematics is inaccessible, and that it has to be done to get through the degree programme. The strong sentiment of avoidance and not wanting to embrace mathematics suggests considerable anxiety towards the subject. The students who presented these themes comprise three males and three females and have MAS-UK scores ranging from 48 to 94, a notably higher range than the students presenting positive themes and suggesting that these students typically have higher levels of mathematics anxiety.

Contrasting themes included:

- A love/hate relationship;
- Wonder and frustration;
- A begrudging respect;
- Not giving up despite being mathematics anxious.

These students expressed an appreciation for mathematics, and acknowledged it importance; however, they struggle with the subject and admit that some of it is very difficult, but they persevere and demonstrate resilience with mathematics. These students comprise three males and one female with MAS-UK scores ranging from 28 to 67.

Two final themes use metaphors to symbolise their feelings towards mathematics. These are ‘Mount Everest’ and ‘Something that is parked there.’ These themes highlight a struggle or challenge with mathematics, as well as the feeling of reluctance to engage with mathematics and that it is not what the student really wants to do, even though it is a requirement. These students are both females and have MAS-UK scores of 61 and 86 respectively.

Discussion

Students demonstrating higher levels of mathematics anxiety tend to have negative attitudes towards mathematics and negativity in respect of their ability in the subject (Ashcraft, 2002). The themes presented in this study reflect a variety of attitudes towards mathematics based on the students’ engagement with the subject throughout their lives. These mature students have shared insights into their mathematics stories and characterised these insights using sometimes evocative language. The themes presented in the title – divorce, evil and the regime of terror – reflect harsh backdrops to the students’ engagement with mathematics, as their respective explanations have conveyed. In each case the mathematics support facilities at the higher education institutions have been instrumental in the students’ success in mathematics in higher education. Exposure to different uses of mathematics in the workplace has also contributed to them seeing the relevance of mathematics.

Among the other themes presented, the positive themes demonstrated an awareness of the importance of mathematics as a subject and its significance to the students’ programmes of study, and these were matched with relatively low levels of mathematics anxiety. In contrast the negative themes illustrated the complexity of mathematics for some students, and a sense of struggling with the subject, enhanced by the use of negative connotations such as dislike, unfriendliness, fear and avoidance. For some students there is an admittance that they do not want to do mathematics but have to in order to fulfil their programme requirements. These negative feelings reflect the relatively higher levels of mathematics anxiety. The remaining themes demonstrated contrasting feelings towards mathematics, illustrating a sense of awareness of the importance of mathematics, while being apprehensive about the subject and persevering with it. The use of metaphors also encapsulated the sense of difficulty that some students attach to mathematics.

Mature students overcome many challenges in order to engage effectively with their undergraduate programmes of study. Service mathematics is – for many mature students – one of these challenges (Fitzmaurice et al., 2014). The mature students that participated in this study have presented various levels of mathematics anxiety as facilitated through the MAS-UK test, and through their interview contributions have talked about diverse challenges posed by mathematics in different ways; some of these mature students embrace these challenges, actively seeking help, while having an appreciation of the benefit and relevance of the subject to their future careers. For others, engagement with mathematics may represent the biggest academic challenge they have to face, and they do so with mixed feelings, comprising feelings about past experiences, as well as a motivation not to let their difficulties with mathematics have a detrimental effect on them.

Conclusion

Having interviewed these twenty mature students, the researcher has been exposed to stories of how these individuals have experienced mathematics from their earliest days at school to the present day. Their mathematics life stories have emphasised the importance of those early experiences in shaping their life's engagement with mathematics, including interactions with significant people, such as teachers and parents, in respect of mathematics (Ryan & Fitzmaurice, 2017). In addition, the mathematics life stories of these mature students have heightened the awareness of the author to those issues conducive or detrimental to effective learning of mathematics. Negative experiences – sometimes stemming from a repeated lack of understanding of concepts – can impact severely on a student's confidence and self-esteem around mathematics, sometimes scarring them in respect of subsequent engagement with the subject.

Fortunately, as these students have revealed, with adequate support and coping strategies adversity towards mathematics can be tackled and overcome leading to success in mathematics (Safford-Ramus, 2008; Fitzmaurice et al., 2014); to this end the varied mathematics support services provided by higher education institutions play an important role (Lawson et al., 2003; Fitzmaurice et al., 2014) especially to those students struggling with mathematics as a consequence of mathematics anxiety. In addition, allowing students the space to talk about their – particularly negative – mathematics experiences (Tobias, 1993), can contribute to the healing process of those scars left behind from negative experiences. In this regard the mathematics life story interview serves a valuable purpose in helping mature students with any level of mathematics anxiety elicit their experiences and identify the challenges that mathematics poses to them as mature students.

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Adults Learning Mathematics: transcending boundaries and barriers in an uncertain world

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Abstract

Adults return to study mathematics for a range of varied and complex reasons. This article addresses the boundaries and barriers that adult learners of mathematics may have faced in the past, or in the present and the future. Taking into account the impact of our era of rapid technological change, I argue for the importance of developing structural understanding of disciplinary mathematics together with the equally important contextual knowledge and skills necessary for understanding and engaging with mathematics in often vastly different circumstances from the past. The goal is for adult learners to feel confident, socially included, and able to participate in the mathematical activities with which they are confronted over a lifetime.

Introduction

Adults return to study mathematics for a range of varied and complex reasons. Although some programs focus on preparing for entrance into specific professions or vocations, other more general programs often aim at meeting economic and/or social inclusion goals. In many countries formal mathematics curricula have given way to sets of learning outcomes where the focus is on what learners can do rather than what they know and how they know it. One consequence is that adult numeracy education (as it is commonly known) consists of collections of mathematical tasks which necessarily have obvious links to practical applications. In vocational education this can often take the form of solely those tasks that are deemed to be of immediate use and easy to demonstrate. Often for lack of available time, this is likely to come at the expense of developing depth of mathematical knowledge and understanding.

A second major issue globally is that curricula and learning outcomes in general tend to focus on past and present economic and societal demands, even though these are both undergoing massive change as exponential rates of technological advance reconfigure how people live and how they will work, or not, as the case may be. In this article my aim is argue the case for ensuring that the learners (who have often missed out in earlier years) develop meaningful structural understandings of number, space, and shape that will enable them to continue to build on their mathematical (and other) learning throughout a lifetime, and to understand the mathematical communications of others at work and in civic life generally.

In this article, I will focus on adults, on learning, and on mathematics, separately and collectively. Focusing on adults, I will address briefly the possible interests that they might have in learning mathematics, particularly in a fractured and fragmented world with constantly changing horizons in terms of politics, economics, technology, the environment, and so on. Focusing on mathematics, I will identify three different perspectives on mathematics, briefly summarise what doing mathematics means at work, and consider the importance of mathematics education in relation to democracy. Focusing on learning (set in the context of the adult world of work), I will address the need for workplace based innovation (as well as standardisation), and the importance of informal learning at work. I will draw on Bernstein's theories to stress the importance of understanding the big ideas of mathematics, and hence its underlying structures and relationships, as well as the crucial importance of contextual knowledge. Finally, in order to support the kind of mathematics education necessary in this era of globalisation and rapid change, bringing about uncertainties to a degree not previously experienced in human history, I will draw parallels between knowing in the discipline of mathematics and in the discipline of music.

Adults: what are their interests in learning mathematics?

In this section I will survey major reasons, past, present, and future, for adults returning to study mathematics, highlighting some of the barriers and boundaries they have confronted in the past, and outlining those which the future might hold.

The past: unfinished business

The first two parts of this section draw largely on FitzSimons (1994). For adults, unfinished business may arise from restricted access to mathematics education, or education more generally, due to a variety of reasons which may be broadly classified as socio-economic, socio-political, and cultural-historic, but which intersect and overlap, and are in a constant state of flux.

In the formative years, a variety of contextual situations may lead to a child missing school or changing schools frequently, or leaving school at or below the minimum age, perhaps even prematurely to their desires and/or abilities. These include relative poverty, chronic illness or unplanned pregnancy during the school years, social and cultural traditions and expectations (particularly in the case of girls and women), belonging to a social or cultural minority group, and family immigration choices. Deliberate acts of human aggression and/or destruction and cultural discrimination (race, gender, religion, etc.) also have potentially negative effects on children's education. Such acts include views of individual worth vs. political expediency (at all levels of education), outdated beliefs in conservatism (back-to-basics) and control in education (curriculum, pedagogy, & assessment), political preferences for funding favoured groups (classes, cultures, religions, etc.) while de-funding others, individual teachers' negative discrimination/mistreatment (mental, physical, racial, sexual, etc. abuse). On a more global scale, children's education may be disrupted or non-existent due to the impact of war, ethnic cleansing, and forced relocations; disruptions also occur as a result of environmental disasters, natural as well as due to human activity.

The present: more contemporary reasons

In addition to the desire to overcome and ameliorate prior disadvantage, as discussed above, adults may find themselves motivated to return to study mathematics for a variety of reasons, according to the circumstances they face at the time or see as likely in the future. As adults they are likely to have more agency and control over some things, but not all. The broad classifications listed above (social, cultural, economic, and political) may still apply.

From personal experience, one of the most important reasons for adults choosing to return to study mathematics is likely to be curiosity: the joy of wanting to learn (more) mathematics — its big ideas, techniques, puzzles, history, aesthetics, etc. — for its own sake, and now finding it actually making sense! Seeking personal enrichment and new friendships, and being among a group of like-minded adults who are sharing this journey, affectively (Kelly, 2018) and cognitively, in a respectful environment where no questions are belittled or ignored — no matter how seemingly oblique or "dumb," and encouragement is pervasive — are motivations among adult learners. The fact that there are alternative, meaningful, and respectful adult teaching approaches, accepting that there are many points of view and ways of looking at problems which are frequently derived from the adults' own experiences, also encourages people to join and remain with the class. Helping others, such as children, and other family members, also community groups (e.g., children's homework, a family business, social or political groups) is often given as a reason for joining a class. There might also be an expressed need to keep up with technology, especially new mathematical and other learning technologies, or to (re)learn mathematics in English or other new home language (see, e.g., Safford-Ramus, Maaß, & Süss-Stepancik, 2018).

Very often, the desire to study mathematics arises from a re-evaluation of the learner's current situation. It could be finding a new direction, often after a significant life change, including relationship breakup, children all left school/home, release from carer's duties, etc. It could be due to the dawning realization of having had limited options over a lifetime or in current work, for example, and attempting to overcome these. For many people at this point, the need (but not necessarily the desire) to study mathematics could arise as a compulsory requirement: as a condition for receiving government or other benefits, or as an entry test or hurdle requirement for employment and/or further education. Sometimes awareness of the importance of being able to do more mathematics, and to feel comfortable in using it, arises part-way through, or even after, programs delivered in or for the workplace, as people feel more empowered to negotiate, to answer back to employers' claims of poor work practice based on incorrect or misleading statistical claims. On a more positive note, workers feel more empowered to ask intelligent questions about work practice with a view to improvement or the avoidance of serious errors (FitzSimons, 2014, 2015; FitzSimons & Boistrup, 2017; Kelly, 2018).

Contemporary and future issues: globalisation and technology

Many of the issues listed above are related to the phenomenon of globalisation which has, by its very nature, changed the conditions of living and working (in the industrialised world, at least) over a relatively short space of time.

The phenomenon of globalisation has been characterised as a series of inter-related flows, including knowledge, capital, jobs, people and cultures, ecological systems. It is also associated with increasingly rapid developments in information and communication technologies [ICTs] as well as transport, facilitating cultural and economic exchange. (CIEAEM, 2010)

Underlining the massive changes in the way that people have lived and worked, Gray (2016) reminds us that the first Industrial Revolution used water and steam power to mechanize production, the second used electric power to create mass production, and the third used electronics and information technology to automate production. He claims that the fourth Industrial Revolution (where we now find ourselves) builds on the third, the digital revolution that has been occurring since the middle of the last century.

Discussing the fourth Industrial Revolution from a human capital perspective, Gray (2016) argued that there will be major changes to the skills that have, until recently, been considered important in the workforce. Technological developments which transcend national boundaries are transforming the way people live and the way they work. This means that some jobs will disappear, others will grow, and jobs that don't even exist today will become commonplace. As a consequence, creativity will become one of the top three skills workers will need to accommodate new products, new technologies, and new ways of working. People working in sales and manufacturing will need new skills, such as technological literacy. The likelihood is that there will be several changes of jobs and even occupations, in different geographical locations, periods of unemployment or under-employment, and either more casualised/contract labour or relatively more secure employment for the fortunate, with a hollowing out at the centre. Automation and the technologically mediated global movement of people, knowledge, and capital, is profoundly changing work and working lives. The demand for highly skilled workers has increased while the demand for workers with less education and lower skills has decreased.

By 2020 the top three skills needed by workers will be:

- Complex problem solving
- Critical thinking
- Creativity (Gray, 2016).

This global shift towards the ongoing development of advanced technologies in work and elsewhere has implications for education, particularly for adults returning to study mathematics for work related reasons. They are likely to be involved in various forms of education throughout their lifetimes, while having to adapt to ongoing technological change in the mathematics classroom and beyond. No longer is it sufficient to equip adults with just the so-called “basic skills,” important as they are. Adults need to understand the structural properties of number and quantities, right from the beginning (Tuominen, Andersson, Boistrup, & Eriksson, 2018), as I will discuss later.

On a personal note, I grew up in an age of apparent certainty, change happened at much slower rate, jobs were for life, and vocational qualifications or a basic university education was for most people considered the completion of an education. Lifelong learning was not even considered as a possibility or necessity, technology was in its infancy, and school subjects were compartmentalised; learning mathematics had little or no obvious relation to any other school subject or everyday life, apart from number skills and measurement.

As a mathematics teacher over many decades, and across different sectors of education, I have had to learn new content, new pedagogies, and new techniques. In terms of mathematical content, I was able to do this relatively easily because I had a sound understanding of the basic structures of mathematics. Any teacher or researcher starting in a new position will also realize just how much they need to learn about the context: “how things are done around here.” These ideas about the need for mathematical structures and contextual knowledge will be elaborated on below.

Considering that the fourth Industrial Revolution is already upon us, and likely to intensify in ways we cannot as yet conceive, how can we as members of ALM, as mathematics (& statistics) teachers in adult, vocational, and further education, and as researchers, support our students to prepare for learning new ways of living and working (just as people of my generation have had to learn about and to participate with some degree of confidence in the rapidly evolving technological world)? This means not only preparing for participation in the economic sphere, but also in the aesthetic and cultural spheres where mathematics plays an integral role as part of our humanity.

Perspectives on mathematics

The word mathematics is often taken for granted. In this section I offer a brief selection from the literature to illustrate the complexity of the concept of mathematics, going beyond the relatively narrow impression given by most school mathematics texts and public discourse, thereby setting implicit boundaries as to what is valued in society.

A cognitive perspective

According to David Tall (2013, see chapter 6), mathematical thinking develops through three worlds of mathematics:

- the practical mathematics of shape and number, with experiences in shape, space, and arithmetic (embodied operations);
- theoretical mathematics with a focus on properties, leading to Euclidean proof and algebra, and definitions based on known objects and operations; &
- formal mathematics, working with properties proved from given axioms and definitions, and formal objects based on formal definitions.

Tall stresses that each world is interlinked with the others and valuable in its own right.

Although invisible in most workplaces, the logic and mathematical power of formal mathematics underpin technologies of management and production: for example, extending human capabilities in the realms micro- and nano-technology, extreme speeds, temperatures, distances, etc. The more easily visible theoretical mathematics, with its symbolic algebra and geometry, can be seen in the functioning and design of spreadsheets, three-dimensional machining, and quality control statistics, for example. Finally, the conceptual embodiment of shape and space, the operational symbolism of arithmetic, with ubiquitous forms of measurement (formal and informal) are the most easily visible forms of practical mathematics (FitzSimons & Boistrup, 2017). Although practical mathematics is where we begin as humans, it is also an essential part of much mathematical activity at work and elsewhere to ensure that desired outcomes may actually be achieved. One important aspect of doing mathematics is accurate communication.

A sociocultural perspective

Anna Sfard (2017, p. 42), in her conceptualisation of mathematics "as a certain well-defined form of communication," asserts that: "to think mathematically means communicating — with others or with oneself" — in the special way generally endorsed by the mathematical community. Her major contribution to the field is that cognition and communication are fully integrated, and she coined the word commognition to capture this.

A philosophical perspective

Roland Fischer (1993) provides an explanation for the apparent alienation of people from mathematics on a personal level when he outlines the duality of mathematics as a means and a system:

> Mathematics provides a means for individuals to explain and control complex situations of the natural and of the artificial environment and to communicate about those situations.
> Mathematics is also a system of concepts, algorithms and rules, embodied in us, in our thinking and doing; we are subject to this system, it determines parts of our identity. This system runs from everyday quantifications to elaborated patterns of natural phenomena to complex mechanisms of the modern economy. (pp. 113-114)

It is the systemic, embodied, side of mathematics which is often overlooked in public discourse. However, both aspects have a role to play in people's lives, and mathematics education plays a crucial role in helping learners of all ages to develop agency. (See FitzSimons, 2002, for further discussion on mathematics and mathematics education.)

Doing mathematics at work (paid & unpaid)

Doing mathematics at school and doing mathematics at work are two very different activities, epistemologically and socioculturally (FitzSimons, 2013). Most mathematics teachers and students who visit work sites, and even workers themselves, find it very difficult to recognize any activity they are able to judge as being mathematical beyond number and measurement. At work, according to FitzSimons and Boistrup (2017):

> people are required to use, develop, and communicate mathematical ideas and techniques in a diversity of ways with others who have differing expertise, experience, and interests, including in mathematics itself. Problems requiring mathematical reasoning and calculations are usually embedded in physical and/or intellectual tasks, rich in context, with a range of constraints that are oftentimes mutually contradictory, but always need a workable answer & within a short space of time. ... In many jobs problem solving is an expected and routine part of the day's work: Every new request or order requires an original or customised solution within given parameters. Whether using mathematics explicitly or implicitly in these processes, no matter how trivial, the worker must also take into account all of the relevant contextual knowledge in their decision making. Crucially, the kinds and complexity of problems that occur at work contrast sharply with those found in formal mathematics education texts and assessment tasks. (pp. 229-230)

One of the most important aspects of working life is the continuous learning that takes place, mostly unconsciously and generally unremarked. First, however, I briefly address pedagogic rights.

Democracy and mathematics education

Bernstein (2000) advocates three pedagogic rights which enable democracy to function: enhancement, inclusion, and participation. At an individual level, the right to enhancement is the right to the means of critical understanding and to new possibilities and a condition for confidence. At the social level, is the right to be included, socially, intellectually, culturally, and personally (inclusion). At the political level, the right to participate must be about practice which has outcomes (participation). How can adult mathematics education position itself to support students in asserting these three rights, so essential in this uncertain world of rapid economic, social, and technological change?

Bernstein (2000) discusses formal mathematical pedagogic situations, where arbitrary selections are made from the discipline of mathematics which provide different forms of consciousness to different social groups, with differential access to what he terms unthinkable knowledge in the possibility of new knowledge creation (e.g., curiosity, creativity, innovation), as distinct from thinkable knowledge which takes the form of official knowledge (e.g., rule following). Given that both forms of mathematical knowledge, thinkable and unthinkable, are likely to support adults developing agency in work and life generally, in line with the future trends outlined above, I turn to the kind of mathematics education which encompasses both kinds of knowledge development.

Perspectives on learning

In this section I will focus on the kinds of learning that can take place in adults' lives, where their mathematical knowledge and skills can play a crucial role in supporting others as well as in enhancing their own agency in our era of rapid change and uncertainty.

Learning through practice-based innovation

Learning takes place throughout a lifetime, consciously or otherwise. As discussed above, adults today will have a much greater need to learn than in generations of the past, especially in relation to the need for innovation. In the workplace, Ellström (2010) identified practice-based innovation as a cyclical process of learning emanating from the problems that need to be addressed and solved on a daily basis. The need for innovation may occur within a logic of production, where speed, accuracy, and consistency are needed and valued in routine procedures and processes — for example in standardised medical and scientific procedures or manufacturing production processes. Starting from the explicit work process, officially prescribed and based on existing codified knowledge (theories, models, etc.), following the logic of production there is a process of adaptive learning and reproduction which results in the implicit work process, where the work is subjectively interpreted and performed on the basis of tacit knowledge. However, where possible legally and in practice, variation and improvisation play a role leading to developmental learning, and the task is transformed through what Ellström termed the logic of development, where questioning, creativity and innovation are more highly valued. As these innovations are accommodated into explicit work practice, the cycle continues. The relationship with Gray's (2016) top three skills of Complex problem solving, Critical thinking, and Creativity is clearly apparent here. In many jobs, problem solving is an expected and routine part of the day's work, where each new "task" demands individualised attention to specific contextual setting and "customer" request, whether arising from within or beyond the actual worksite.

Informal learning at work

As noted above, many people do not consciously recognize that learning takes place at work. Learning via non-formal education may occur via specific activities such as training courses in new procedures or new skills development. Informal learning at work is an ongoing process that encompasses personal, social, and cultural knowledges and skills (Eraut, 2004). Importantly, there is an ongoing need for communication of an educative kind between stakeholders where information, often mathematical, is sought and shared (see, e.g., FitzSimons, 2013). Such communication can be considered as a pedagogic relation (Bernstein, 2000). People's mathematics-related knowledge is, or can be, transformed from the academic discipline of mathematics to address the specific context of the problem at hand. Pedagogic relations occur frequently in communications at work, for example between: (a) co-workers, (b) managers or team leaders and workers, (c) clients and their suppliers of services or goods, etc. Workplace mathematics is not commonly recognized as a pedagogic activity, unlike school mathematics where the activity takes place within recognized spaces, at particular times, for given periods, with a recognizable hierarchy between the teacher and the taught. School mathematics curriculum and assessment is generally controlled by external authorities, and conducted via a limited range of activities, mostly verbal and written, sometimes online, and sometimes with specific mathematical tools such as rulers and protractors. By contrast, mathematics at work may be partially or completely structured by the available artefacts and employing a range of verbal and non-verbal communication modalities (Björklund Boistrup & Gustafsson, 2014), and takes place within a wide range of possible constraints, such as time, money, specific industrial legal requirements, and occupational health and safety laws. Most important, mathematical solutions at work need to be fit for the particular purpose and error free.

Pedagogic relations at work

How do pedagogic relations at work actually function? Following Hordern (2014), recontextualisation offers a powerful means of "bridging the gap" between theoretical and experiential knowledge. Recontextualisation combines the disciplinary knowledge of mathematics with contextual knowledge of the specific situation, including personal and sociocultural knowledges, in the form of communication (with self or others; cf. Sfard, 2017). Adult students in formal education also need to learn these skills to communicate effectively with other stakeholders. Doing mathematics at work (& elsewhere) involves: learning to find out what to do, how to do it, when, where, how often, and sometimes why, ... until there are no more questions. However, the learning and teaching process is two-way, depending on who has information and/or authority to speak, to ask, and to answer (FitzSimons & Boistrup, 2017). Communicating at work can take place through symbols, diagrams, speech, gestures, and so on — often simultaneously, described by Björklund Boistrup and Gustafsson (2014) as multimodal. In summary, (mathematical) recontextualisation at work involves transforming what you know, both mathematically and contextually, to make the most appropriate decision; or perhaps even to ask further questions.

The importance of knowledge structures

In order for adults to be able communicate effectively at work and elsewhere, what kind of mathematical knowledge structures do they need? Is it enough to teach them only "the skills they need" — as seems to be common practice in many English-speaking countries around the world where competency-based frameworks hold sway over adult literacy and numeracy as well as vocational education? Random collections of mathematical rules and 'tricks' needed for particular contexts will not be sufficient for learners/workers to develop new knowledge to meet local conditions unless they have access to the integrating structures of the discipline of mathematics as a foundation on which to build.

The importance of both vertical discourse and horizontal discourse

Bernstein (2000) describes vertical discourses such as mathematics as being comprised of disciplinary knowledge (albeit fallible), theoretical, conceptual, and generalizable knowledge, coherent, explicit, and systematically principled. Because the procedures of vertical discourse are linked hierarchically, they allow the integration of meanings beyond relevance to specific contexts. In other words, once you understand a mathematical principle, you can use it in a range of contexts. However, although a working understanding of the vertical discourse of mathematics (at whatever level the person has reached) is essential, it is not enough to guarantee that a person will be able to communicate and participate fully at work or as a citizen. Knowledge of the horizontal discourse within the particular context is also absolutely necessary to make sense of the task at hand.

According to Bernstein (2000), horizontal discourse refers to contextual knowledge, which is likely to be oral, … tacit, multi-layered, and contradictory across but not within contexts (p. 157). Individual skills may have nothing in common with other skills, but may be related to specific contexts or to everyday life, and pedagogic practices for these may well vary with each activity. Examples include learning to tie up one's own shoe laces, brush one's teeth, ride a bicycle, drive a car, make a cup of tea, ... Such knowledges are "culturally localised, and evoked by contexts whose reading is unproblematic" (p. 159). (As an aside, these kinds of skills may be lost with the degeneration that occurs with old age!) A person may build up an extensive repertoire of strategies which may vary according to the context; a group may likewise build up a reservoir of strategies of operational knowledges. In horizontal discourse, there is not necessarily one best strategy relevant to any particular context. (See FitzSimons & Boistrup, 2017, for further explanation in relation to mathematics at work.)

Being competent to work within the vertical discourse of mathematics is necessary but not sufficient. When employers complain that new graduates cannot use their mathematical knowledge and skills, this is because they have not yet comprehended the horizontal discourse at the local worksite. There are specific ways of being a research scientist, a mathematics teacher, a plumber, etc., and it is also crucial for learners and workers to develop the necessary situational and contextual knowledge and skills (practical, personal, social, structural, etc).

In order to illustrate the claims made above in relation to knowledge structures, I turn to the theoretical discipline and practice of music which in many ways bears a parallel to mathematics but may offer insights through the potential lack of familiarity (or "otherness") to many readers belonging to the adult mathematics education community.

Mathematics and music

Structured bodies of knowledge are important. They allow us to account for and explain the natural and social world in systematic ways, and to participate in, and reflect on key human experiences (see Bernstein & Democracy, above) such as in the literary, visual, or musical domains. Disciplines such as mathematics and music enable abstraction, reflection, prediction, and application across time and contexts.

At ALM conferences, music has played a big role over the last 25 years in supporting social cohesion and in breaking down potential boundaries between the many international participants in an enjoyable and instructive way. Many conference delegates have revealed hitherto unknown talents in musical and theatrical performance. (This was particularly in evidence at the ALM 25 conference dinner!) The question is: How is it that people can have agency to learn and to create new music beyond merely learning by imitation — important as that is in passing on cultural knowledges and skills?

Music and mathematics have many similarities. Music performance, like mathematics, requires mastery of simple pieces, gradually building up to a reflective, nuanced, and more original practice.

Formal music education, theoretical and practical, offers insight into musical conventions and boundaries, even if only in order to violate them! Mathematics and music are both pan cultural, with historical and cultural variations, including ethno-mathematics or ethno-music, and are available to all of us as part of our humanity, in theory at least: for example, rhythmic clapping and community singing. Mathematics and music each have: (a) a structured body of knowledge which is ever evolving, (b) a codified set of symbols with universally accepted meanings within the discipline, and (c) several different sub-fields (instrumental & vocal specialisations, genres etc. in music; specialisations such as algebra, geometry, statistics, etc. in mathematics), each with their own rules, but within the framework of a vertical discourse. In an era where integrated STEM [Science, Technology, Engineering, & Mathematics] education is widely promoted, in some form of interdisciplinary education, there is no reason why music should not be included — perhaps as a form of technology. In FitzSimons (2001) I offer an example of my own experience of integrating disparate disciplinary structures in workplace mathematics and statistics education.

Music, like mathematics, also has the possibility of knowledge and skills leading to performance being passed on through an oral tradition, often culturally specific, and displayed/performed in contextually appropriate settings. In music, popular social and cultural songs are sung on appropriate occasions, and are a valuable means of social cohesion, for example: Happy Birthday, Auld Lang Syne, Danny Boy (Londonderry Air), Will Ye Go Lassie Go? In mathematics, most children have the opportunity to learn counting rhymes, songs, and related stories. Both vertical and horizontal discourses exist in music and mathematics: knowing what to do, how to do it, where, why, and when.

As in mathematics, there are obvious differences between professional, amateur, and novice musicians. The more skilled and knowledgeable the musician, the easier it is for them to confront and solve unexpected problems that may arise in the form of being asked to sight-read and perform an unseen piece of music, or even to perform it in a different key (transpose it up or down). When you are confronted with a piece of sheet music, what do you see? Does it have any meaning for you? What meaning you make depends on where you are on the continuum of expertise. Is it too much of a stretch to imagine that the task of interpreting an unfamiliar page of music is parallel to how mathematics appears to many adults? What are the consequences for people with a limited, interrupted, or discontinuous mathematics education? Do they feel confident, feel included, and are they able to participate in this human activity? How will learning only “the maths you need” for a certain job at a certain moment in time help people to live in the world in our uncertain times?

A brief example

In work I did recently to support young adult learners returning to study mathematics and their tutors and in a numeracy program situated in a learning outcomes framework, the Measurement outcome stated:

> Can use straightforward measurement and the metric system to estimate and measure for the purpose of interpreting, making or purchasing materials in familiar practical situations.

It contained three Elements: Mathematical knowledge and techniques, Language, and Interpretation. These were elaborated using active verbs such as estimate, choose, use, read, convert, calculate, interpret, and decide on reasonableness.

At first glance, these sound reasonable, and the document offered possibilities for holistic assessment. However, there are several things to observe. First, the language implies rule following, and clearly measurement should be appropriate, and accurate to appropriate levels. Although it is important that these learners should be familiar with parts of the metric system listed, the document did not invite them to pursue related mathematical questions of their own, and it offered no intimation of a holistic system of measurement, the Système Internationale or SI. It is likely that some learners in the group had not yet developed a secure understanding of the decimal number system, or even the basics of whole numbers, and none of the small number of professional staff had any post-school mathematics qualifications. There was no underpinning structure of the mathematics available to assist the teaching staff in identifying predictable knowledge gaps or to demonstrate to staff and learners how the various mathematical outcomes fit together. The selection of a collection of well known, but not necessarily well understood, metric units and prefixes limits the ability of learners to see the bigger picture and develop their own knowledge in this area. As such, it is not clear how learning outcomes such as this could engender the creativity, problem solving, and critical thinking (Gray, 2016) in relation to the learners' mathematical knowledge.

In the limited space here it is not possible to outline a whole teaching program, but here are some possible early approaches in classes of learners with diverse and intermittent histories of schooling (in no particular order). Ideally, the learners should be working in small groups.

- Explore the history of the basic metric units of length and mass, and the use of water in the latter
- Construct a portable "metre ruler" (e.g., from foldable soft cardboard) and, when appropriate, subdivide it into decimetres and centimetres; similarly with common & decimal fractions, and percentages, in order to anchor meanings and build cross-connections
- Explore (together) the origin, etymology, and use of language in the (invariant) metric prefixes, and gradually build up a metric table for each learner or group of learners
- Extend the meanings of prefixes to introduce the concept of powers of 10, eventually using exponential notation when appropriate (e.g., 102)
- Use the internet to investigate very large and very small numbers
- Collect food packaging (e.g., rectangular boxes, cans), and analyse the mathematical information on the labels; also draw out relations between lengths, surface areas, volumes, and capacity
- Problem solve how to deal with irregular shapes
- Develop embodied knowledge about the approximate length of 1mm, 1 cm, 1 metre, etc., also the mass of various common items weighing a gram, 100 gram, 500 gram, a kilogram etc.

On a larger scale, invite learners to participate in projects involving measurement and data where they can achieve concrete outcomes within given mathematical and other parameters; where possible these should be agreed upon by teacher/tutors and learners alike. These could include designing activities and explaining the reasoning to others. Most importantly, these projects offer the opportunity to confront reality with its inherent constraints, and where the best mathematical solution may not be the best in practice or may not even be feasible at all. For example:

- In the local community space, indoors or outdoors, (re)design or renovate an existing space; or organise a theme-based walking tour or fun run for community members.
- Collect, analyse, interpret, and present data in order to argue for or against a case concerning a locally important issue
- Collect information on health & nutrition, keep diaries of relevant food intake over a given period of time, prepare and serve a meal to others in keeping with recommended food guidelines.

It may also be feasible to involve learners in the assessment of such projects, particularly those of other groups (see, e.g., Hahn & Vignon, 2019); even evaluating the project as a whole. Such projects expose the learners to both the vertical discourse of mathematics and the horizontal discourses of the tasks. They also have the potential for learners to notice serious flaws in their mathematical calculations, reasoning, or assumptions in a much more memorable way than textbook learning could ever do. The logic underpinning such activities is to invite the learners to actively participate in the world of mathematics in ways that have meaning for them, and to see mathematics as both a means towards achieving an outcome or to support making a decision, and as something that is part of our common humanity (cf. Fischer, 1993).

Conclusion

Although there are obviously many other theoretical and practical approaches to adult education (see, e.g., FitzSimons, 2010), this article does not attempt to reflect directly on any particular teaching practice. I have drawn predominantly on the work of Bernstein (2000) to argue for the importance of developing structural knowledge in the discipline of mathematics, from the earliest educational levels. At the same time, adults need to be as fully aware as possible of the given contextual setting of the situation they are addressing, in order to be able to recontextualise the mathematics they know in ways that are adequate and appropriate to that situation. Communication and recontextualisation are multi-directional processes to understand and inform the thinking of ourselves and others.

As Bernstein (2000) argues, abstract theoretical knowledge enables society to conduct a conversation about itself, and to imagine alternative futures; and access to “unthinkable” knowledge enables the possibility of new knowledge creation (i.e., through curiosity, creativity, innovation). For a person to have agency, to have confidence, to feel able to be included, to have the right to participate in the discourse of the discipline of mathematics, and to be able to learn new things and to create (locally) new knowledge, they must have access to the most relevant, structurally supported and connected mathematical knowledge, which is at the same time flexible enough to meet their contextually specific demands. This means that they should be able to recognise the same or similar mathematical structures in novel contexts and transform the mathematics they already know, creating a potentially innovative and workable solution in that particular context.

Learning outcomes in adult and vocational education often tend to abstract the lived experiences of adults from the power relations of work and life and to negate the possibilities of understanding and criticism. We need to teach adults how to use mathematics to answer back to power at work and in society more generally, going well beyond the mastery of mathematical techniques and the following of sets of instructions without question. For this reason, to paraphrase the words of one reviewer, we need to be alert to adult mathematics education contexts that undervalue structural mathematics.

In stressing the importance of taking into account technological innovation and rapid change globally, rather than a backward looking orientation on the reproduction of assemblages of skills of the past, I raise the following question: Do we, as adult mathematics education teachers and researchers, not have an ethical responsibility to ensure that our students, at whatever level, develop more secure and connected content knowledge in order for learners to have the possibility of transcending the boundaries and barriers at work and elsewhere in an uncertain world. Without access to both vertical and horizontal discourses, often invisible boundaries and barriers will be very difficult to transcend.

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